

Derivation of Nonlocal Finite Element Formulation for Nano Beams

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Received date: 01.08.2018

Accepted date: 14.08.2018

Abstract

In the present paper, a new nonlocal formulation for vibration derived for nano beam lying on elastic matrix. The formulation is based on the cubic shape polynomial functions via finite element method. The size effect on finite element matrix is investigated using nonlocal elasticity theory. Finite element formulations and matrix coefficients have been obtained for nano beams. Size-dependent stiffness and mass matrix are derived for Euler-Bernoulli beams.

Keywords: Size-dependend vibration, cubic shape functions, Nonlocal elasticity, Euler-Bernoulli beam, new mass and stiffness matrix.

1. Introduction

It is known that the nanotechnology has enabled the opening of a new era in many areas in nano optics, microcomputers and micro devices, chemical, medicine, engineering, electronics. For keep up with technology fast, the correct solution method which considers the size effect is the most important factor. Experimental research is very difficult and expensive. Some methods such as Hybrid atomistic–continuum mechanics and related to the atomic modeling; molecular dynamics [1-3], tight-binding molecular dynamics, the density functional theory take into account the size effect. Therefore, various theories have been developed that gives importance to effects of small scale such as strain gradient theory [4,5], modified couple stress theory [6-9], couple stress elasticity theory [10-13], nonlocal elasticity theory [14-15]. Nonlocal elasticity theory of Eringen is the most widely used among them. According to the nonlocal elasticity theory of Eringen [14-15], the stress at any reference point is effecting the whole body which not depends only on the strains at this point but also on strains at all points of the body. This definition of the Eringen's nonlocal elasticity is based on the atomic theory of lattice dynamics, and some experimental observations on phonon dispersion. Nonlocal



theory considers long-range interatomic interaction and yields to results dependent on the size of a body [14-16]. Applying first the nonlocal elasticity theories to nanotechnology is by Peddieson et al. [17] and Sudak [18]. Nanostructures with nonlocal elasticity theory have been studied for different type (numerical and analytical) solution with contributions continuum mechanics by finite element method [19-25], by finite difference method [26-27] by differential transform method [28-30], by differential quadrature method [31-34], and by analytical solution [35-43].

2. Size dependent formulation

The main equations for a homogenous and isotropic elastic continuum body can be stated as [14,15,16]:

$$\sigma_{ij,j} = 0, \quad (1)$$

$$\sigma_{i,j}(x) = \int_V \alpha(|x - x'|, \chi) C_{ijkl} \varepsilon(x') dV(x'), \quad (2)$$

$$\varepsilon_{ij}(x') = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad (3)$$

where σ_{ij} is the nonlocal stress tensor, ρ is the mass density of the body, u is the displacement vector at a reference point x in the body, $C_{ijkl}(x')$ is the classical (Cauchy) or local stress tensor at any point x' in the body, $\varepsilon_{ij}(x')$ is the linear strain tensor at point x' in the body, t is denoted as time, V is the volume occupied by the elastic body, $\alpha|x - x'|$ is the distance in Euclidean form, λ and μ are the Lamé constants. $\alpha|x - x'|$ is the nonlocal kernel which defines the impact of strain at point x' on the stress at point x in body. The nonlocal constitutive formulation is

$$[1 - (e_0 a)^2 \nabla^2] \sigma_{ij} = C_{ijkl} \quad (4)$$

The displacement components based on the Euler-Bernoulli beam theory may be written as [36-37]:

$$u = -z \frac{\partial w}{\partial x}, \quad v = 0 \quad w = w(x, t) \quad (5)$$

where ‘ w ’ is the transverse displacement. The strain-displacement equations for Euler-Bernoulli beam is given by

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z \frac{\partial^2 w}{\partial x^2}(x, t) \quad \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0 \quad (6)$$

Consider the stress-strain relation for Euler-Bernoulli beam is given by

$$\sigma_{xx} = -Ez \frac{\partial^2 w}{\partial x^2}(x, t) \quad \sigma_{yy} = \sigma_{zz} = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0 \quad (7)$$

According to Eq. (4), the nonlocal stress-strain equations for beam can be written as [36-37]

$$\sigma_{xx} - (e_0 a)^2 \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E \varepsilon_{xx}, \quad \sigma_{yy} = 0, \quad \sigma_{zz} = 0 \quad (8a)$$

The generalized Hamilton's principle has the form

$$\delta \int_0^t [T - (U - W)] dt = 0 \quad (9)$$

The strain and kinetic energies of the classic Euler-Bernoulli beam are equal to

$$U = \frac{1}{2} \int_V \sigma_{xx} \varepsilon_{xx} dV \quad (10)$$

$$T = \frac{1}{2} \rho \int_V \left[\left(\frac{\partial w}{\partial t} \right)^2 \right] dV \quad (11)$$

The work done by the axial compressive force, Winkler foundation modulus (k_w) and Pasternak foundation modulus (k_g) can be expressed as

$$W = \frac{1}{2} \int_0^L \left[(P - k_g) \left(\frac{\partial w}{\partial x} \right)^2 - k_w (w)^2 \right] dx \quad (12)$$

Substitution of Eqs.(10)-(11) into Eq.(9), acquired

$$\int_0^t \left[\int_0^L \rho A \frac{\partial w}{\partial t} \delta \frac{\partial w}{\partial t} dx - \left(-M \delta \left(\frac{\partial^2 w}{\partial x^2} \right) dx \right) + \left((P - k_g) \frac{\partial w}{\partial x} \delta \frac{\partial w}{\partial x} - k_w w \delta w \right) dx \right] dt = 0 \quad (13)$$

When Eq.(13) under the double integral equal to zero under the double integral, differential equations of motion,

$$\frac{\partial^2 M}{\partial x^2} = \rho A \frac{\partial^2 w}{\partial t^2} + (P - k_g) \frac{\partial^2 w}{\partial x^2} + k_w w \quad (14)$$

The nonlocal moment resultants for beam can be obtained via (8a) as

$$M_x - (e_0 a)^2 \frac{\partial^2 M_x}{\partial x^2} = -EI \frac{\partial^2 w}{\partial x^2} \quad (15)$$

Substitution of Eq.(14) in to Eq. (15) leads to

$$M = (e_0 a)^2 \left(\rho A \frac{\partial^2 w}{\partial t^2} + (P - k_g) \frac{\partial^2 w}{\partial x^2} + k_w w \right) - EI \frac{d^2 w}{dx^2} \quad (16)$$

Finally, by substituting Eq.(16) into Eq.(13), we obtained governing equations for nonlocal Euler-Bernoulli beam [21,22,28,34]

$$\rho A \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2}{\partial x^2} \left[(e_0 a)^2 \left(\rho A \frac{\partial^2 w}{\partial t^2} + (P - k_g) \frac{\partial^2 w}{\partial x^2} + k_w w \right) - EI \frac{\partial^2 w}{\partial x^2} \right] - (k_g - P) \frac{\partial^2 w}{\partial x^2} + k_w w = 0 \quad (17)$$

The Euler-Bernoulli beam element is a beam with four degrees of freedom (DOF) and has two end nodes: 1 and 2. The node displacement vector

$$w^e = [w_1 \quad \theta_1 \quad w_2 \quad \theta_2] \quad (18)$$

By multiplying shape function (ϕ) and discretized displacements at nodes ($[w(t)]^{ne}$) of an element we obtain the displacement of element ($w(x,t)^e$)

$$w(x,t)^e = [\phi [w(t)]^{ne}] \quad \dot{w}(x,t)^e = [\phi [\dot{w}(t)]^{ne}] \quad (19)$$

To solve the equations the ‘Hermitian cubic shape functions’ are used. Dimensionless natural coordinate can be stated as below

$$\xi = \frac{2x}{L} - 1 \quad (20)$$

where L is the element length. By using the shape functions (Eq.(21)) and dimensionless natural coordinates (Eq.(20)), the stiffness matrix becomes (ignoring the axial load)

$$K^1 = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \quad (21a)$$

Similarly

$$K^2 = (e_0 a)^2 \frac{(k_g)}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \quad (21b)$$

$$K^3 = \frac{(k_g)}{30L} \begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix} \quad (21c)$$

$$K^4 = \frac{k_w}{420} \begin{bmatrix} 156L & 22L^2 & 54L & -13L^2 \\ 22L^2 & 4L^3 & 13L^2 & -3L^3 \\ 54L & 13L^2 & 156L & -22L^2 \\ -13L^2 & -3L^3 & -22L^2 & 4L^3 \end{bmatrix} \quad (21d)$$

$$K^5 = \frac{(e_0 a)^2 k_w}{30L} \begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix} \quad (21e)$$

Also, the mass matrix can be given as

$$M^1 = \frac{\rho A}{420} \begin{bmatrix} 156L & 22L^2 & 54L & -13L^2 \\ 22L^2 & 4L^3 & 13L^2 & -3L^3 \\ 54L & 13L^2 & 156L & -22L^2 \\ -13L^2 & -3L^3 & -22L^2 & 4L^3 \end{bmatrix} \quad (22a)$$

$$M^2 = \frac{(e_0 a)^2 \rho A}{30L} \begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix} \quad (22b)$$

$$K = K^1 + K^2 + K^3 + K^4 + K^5, \quad M = M^1 + M^2 \quad (22c)$$

Finally, the vibration of Euler-Bernoulli beam can be expressed as

$$\det|K - \omega^2 M| = 0 \quad (23)$$

3. Results

As numerical results, non-dimensional frequency values of boron nitride nanotube with clamped supports at both have been obtained and results listed in Table 1. The results obtained by discrete singular convolution are also given in this table. It is shown that, the frequency values are increased with the increasing value of nonlocal parameter.

Table 1. Dimensionless Frequency values of C-C boron nitride nanotube

$e_0 a/L$	Foundation parameters ($K_w=10, K_g=50$)					
	ω_1		ω_2		ω_3	
	FEM	DSC	FEM	DSC	FEM	DSC
0.0	33.3266	33.3485	78.0638	78.0873	139.862	139.8836
0.1	34.9378	34.9403	74.1135	74.1205	116.255	116.2673
0.2	37.7969	37.8041	69.3394	69.3593	99.5596	99.5701
0.3	40.0275	40.0359	66.8639	66.8704	93.7733	93.7937
0.4	41.4571	41.4691	65.6397	65.6491	91.529	91.5383
0.5	42.3505	42.3712	64.9805	64.9907	90.5076	90.5214

4. Concluding remarks

Experimental studies have shown that the mechanical behaviors of nano-scaled systems are completely different from structures having conventional dimensions (centimeters, meter dimensions). Experimental study on nanostructures and nanostructures is both costly and time consuming. For this reason, by using higher-order elasticity theories, the results have been theoretically tried to be obtained closer and many higher order theories taking the size effect into account have been emerged. In this paper, nonlocal elasticity theory was used to investigate the dimensional effect for nano beams. Using the nonlocal elasticity theory, equation of free vibration of beams on elastic matrix has been obtained. Obtained differential equation is solved using the finite element methods.

Acknowledgements

This study has been supported by The Scientific and Technological Research Council of Turkey (TÜBİTAK) with Project no: 117M495. This support is gratefully acknowledged.

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