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Exponentially Fitted Finite Difference Method for Singularly Perturbed Delay Differential Equations with Integral Boundary Condition

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Abstract

In this paper, exponentially fitted finite difference method for solving singularly perturbed delay differential equation with integral boundary condition is considered. To treat the integral boundary condition, Simpson's rule is applied. The stability and parameter uniform convergence of the proposed method are proved. To validate the applicability of the scheme, two model problems are considered for numerical experimentation and solved for different values of the perturbation parameter, ε *and mesh size,* h. *The numerical results are tabulated in terms of maximum absolute errors and rate of convergence and it is observed that the present method is more accurate and* ε*-uniformly convergent for* $h \geq \varepsilon$ *where the classical numerical methods fails to give good result and it also improves the results of the methods existing in the literature.*

Keywords: Singularly perturbed problems, Delay differential equation, Exponentially fitted operator, Integral boundary condition

Mathematics Subject Classification: 65L11 · 65L12 · 65L20

1. Introduction

A differential equation is said to be singularly perturbed delay differential equation, if it includes at least one delay term, involving unknown functions occurring with different arguments and also the highest derivative term is multiplied by a small parameter. Such type of delay differential equations play very important role in the mathematical modeling of various practical phenomena

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and also widely applicable in the fields such as biosciences, control theory, economics, material science, medicine, robotics etc [1-4]. Any system involving a feedback control almost involves time delay. The delay occurs because a finite time is required to sense the information and then react to it.

Finding the solution of singularly perturbed delay differential equations is a challenging problem. In response to these, in recent years there has been a growing interest in numerical methods on singularly perturbed delay differential equations. In mid-eighties to mid-nineties, Lange and Miura [5] studied a class of boundary-value problems for second-order differential-difference equations in which the highest-order derivative is multiplied by a small parameter and proposed some asymptotic method to approximate the solution of this class of differential equations. In 2002, Kadalbajoo and Sharma initiated the numerical study of such type of boundary value problems [6-12]. In [13], a fitted operator scheme on a uniform mesh is suggested to solve an initial value problem for a class of linear and semi linear first order delay differential equations. Amiraliyev and Cimen [14] proposed a first order uniform convergent fitted finite difference scheme for singularly perturbed boundary value problem for a linear second order delay differential equation with large delay in reaction term. Subburayan and Ramanujam [15] gave an initial value technique to solve singularly perturbed boundary value problem for second order delay differential equation of convection-diffusion problem with large delay.

The standard numerical methods used for solving singularly perturbed differential equation are sometime ill posed and fail to give analytical solution when the perturbation parameter ε is small. Therefore, it is necessary to develop suitable numerical methods which are uniformly convergent to solve this type of differential equations.

In the present paper, motivated by the works of [16], we developed exponentially fitted operator finite difference scheme on uniform mesh for the numerical solution of second order singularly perturbed convection-diffusion equations with delay and integral boundary condition.

Throughout our analysis C is generic positive constant that is independent of the parameter ε and number of mesh points $2N$. We assume that $\bar{\Omega}=[0,2]$, $\Omega=(0,2)$, $\Omega_1=(0,1)$, $\Omega_2=(1,2)$. Further, $\Omega^*=\Omega_1\cup\Omega_2$, $\bar{\Omega}^{2N}$ is denoted by $\{0,1,2,...,2N\}$, $\bar{\Omega}_1^{2N}$ is denoted by $\{1,2,...,N-1\}$, Ω_2^{2N} is denoted by $\{N+1,N+2,...,2N-1\}$.

Therefore, the main objective of this study is to develop more accurate, stable and convergent exponentially fitted operator finite difference method for solving singularly perturbed convection-diffusion problems with integral boundary condition.

2. Statement of the problem

Consider the following singularly perturbed problem

$$Ly(x)=-\varepsilon y''(x)+a(x)y'(x)+b(x)y(x)+c(x)y(x-1)=f(x),\quad x\in\Omega=(0,2) \tag{1}$$

$$y(x)=\phi(x),\quad x\in[-1,0], \tag{2}$$

$$Ky(2)=y(2)-\varepsilon\int_0^2 g(x)y(x)dx=l, \tag{3}$$

where $\phi(x)$ is sufficiently smooth on $[-1,0]$.

For all $x \in \Omega$, it is assumed that the sufficient smooth functions $a(x), b(x)$ and $c(x)$ satisfy $a(x) > \alpha_1 > \alpha > 0,\ b(x) \geq \beta \geq 0,\ c(x) \leq \gamma \leq 0,$ and $\alpha + \beta + \gamma > 0.$

Furthermore, $g(x)$ is non-negative and monotonic with $\int_0^2 g(x)dx < 1.$ The above assumptions ensure that $y \in X = C^0(\Omega) \cap C^1(\Omega) \cap C^2(\Omega_1 \cup \Omega_2).$

The Eqs. (1)–(3) is equivalent to

$$Ly(x) = F(x) \tag{4}$$

Where

$$Ly(x) = \begin{cases} L_1 y(x) = -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x), & x \in \Omega_1 = (0,1) \\ L_2 y(x) = -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1), & x \in \Omega_2 = (1,2) \end{cases} \tag{5}$$

$$F(x) = \begin{cases} f(x) - c(x)\phi(x-1), & x \in \Omega_1 \\ f(x), & x \in \Omega_2 \end{cases} \tag{6}$$

with boundary conditions

$$\left. \begin{array}{l} y(x) = \phi(x),\ x \in [-1,0], \\ y(1^-) = y(1^+),\ \ y'(1^-) = y'(1^+), \\ Ky(2) = y(2) - \varepsilon \int_0^2 g(x)y(x)dx = l, \end{array} \right\} \tag{7}$$

3. Properties of continuous solution

Lemma 3.1: (Maximum Principle) Let $\psi(x)$ be any function in X such that $\psi(0) \geq 0,\ K\psi(2) \geq 0,\ L_1\psi(x) \geq 0, \forall x \in \Omega_1, L_2\psi(x) \geq 0, \forall x \in \Omega_2,$ and $[\psi'](1) \leq 0$ then $\psi(x) \geq 0,\ \forall x \in \bar{\Omega}.$

Proof: Define the test function

$$s(x) = \begin{cases} \dfrac{1}{8} + \dfrac{x}{2}, & x \in [0,1] \\ \dfrac{3}{8} + \dfrac{x}{4}, & x \in [1,2] \end{cases} \tag{8}$$

Note that $s(x) > 0,\ \forall x \in \bar{\Omega},\ Ls(x) > 0,\ \forall x \in \Omega_1 \cup \Omega_2, s(0) > 0,\ Ks(2) > 0$ and $[s'](1) < 0.$

Let $\mu = \max\left\{\frac{-\psi(x)}{s(x)} : x \in \bar{\Omega}\right\}$. Then, there exists $x_0 \in \bar{\Omega}$ such that $\psi(x_0) + \mu s(x_0) = 0$ and $\psi(x) + \mu s(x) \geq 0,\ \forall x \in \bar{\Omega}.$ Therefore, the function $(\psi + \mu s)$ attains its minimum at $x = x_0$. Suppose the theorem does not hold true, then $\mu > 0.$

Case (i): $x_0 = 0$

$$0 < (\psi + \mu s)(0) = \psi(0) + \mu s(0) = 0, \quad \text{It is a contradiction.}$$

Case (ii): $x_0 \in \Omega_1$

$$0 < L(\psi + \mu s)(x_0) = -\varepsilon(\psi + \mu s)''(x_0) + a(x_0)(\psi + \mu s)'(x_0) + b(x_0)(\psi + \mu s)(x_0) \leq 0$$

It is a contradiction.

Case (iii): $x_0 = 1$

$$0 \leq [(\psi + \mu s)'](1) = [\psi'](1) + \mu[s'](1) < 0, \text{ It is a contradiction.}$$

Case (iv): $x_0 \in \Omega_2$

$$0 < L(\psi + \mu s)(x_0) = -\varepsilon(\psi + \mu s)''(x_0) + a(x_0)(\psi + \mu s)'(x_0) + b(x_0)(\psi + \mu s)(x_0) \\ + c(x_0)(\psi + \mu s)(x_0 - 1) \leq 0, \text{ It is a contradiction.}$$

Case (v): $x_0 = 2$

$$0 \leq K(\psi + \mu s)(2) = (\psi + \mu s)(2) - \varepsilon\int_0^2 g(x)(\psi + \mu s)(x)dx \leq 0, \text{ It is a contradiction.}$$

Hence, the proof of the theorem.

Lemma 3. 2: (Stability Result) The solution $y(x)$ of the problem (1)–(3), satisfies the bound

$$|y(x)| \leq C\max\left\{|y(0)|, |Ky(2)|, \sup_{x\in\Omega^*}|Ly(x)|\right\}, \quad x \in \bar{\Omega}$$

Proof: For the proof refer [16]

Lemma 3.3: The bound for derivative of the solution $y(x)$ of the problem (1)-(3) when $x \in \Omega_1 = (0,1)$ is given by

$$\left|y^{(k)}(x)\right| \leq C\left(1 + \varepsilon^{-k}\exp\left(\frac{-\alpha(1-x_j)}{\varepsilon}\right)\right), \quad k = 0 \leq k \leq 4, \ \ j = 1, 2, ..., N-1.$$

Proof: For the proof refer [17]

4. Formulation of the numerical scheme

For small values of ε, the boundary value problem (1)–(3) exhibits strong boundary layer at $x = 2$ and interior layer at $x = 1$ (see [16]) and cannot, in general, be solved analytically because of the dependence of $a(x), b(x)$ and $c(x)$ on the spatial coordinate x. We divide the interval $[0, 2]$ into $2N$ equal parts with constant mesh length h. Let $0 = x_0, x_2, ..., x_N = 1, x_{N+1}, x_{N+2}, ..., x_{2N} = 2$ be the mesh points. Then we have $x_i = ih,\ i = 0, 1, 2, ...2N.$ If we consider, the interval $x \in (0,1)$ and the coefficients of (1) are evaluated at the midpoint of each interval, then we will obtain the differential equation

$$\begin{cases} -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x) - c(x)\phi(x-1), & x \in \Omega_1 = (0,1) \\ y_0 = y(0) = \phi(0) \end{cases} \tag{9}$$

Now, the domain $[0,1]$ is discretized into N equal number of subintervals, each of length h. Let $0 = x_0 < x_1 < ... < x_N = 1$ be the points such that $x_i = ih,\ i = 0,1,2,...,N$. For the discretization, we apply a exponentially fitted operator finite difference method (FOFDM).
From (9) we have

$$-\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = F(x), \quad x \in \Omega_1 = (0,1) \tag{10}$$

where $F(x) = f(x) - c(x)\phi(x-1)$.
To find the numerical solution of (10) we use the theory applied in asymptotic method for solving singularly perturbed BVPs. In the considered case, the boundary layer is in the right side of the domain i.e. near $x = 1$. From the theory of singular perturbations given by O'Malley [18] and using Taylor's series expansion for $a(x)$ about $x = 1$ and restriction to their first terms, we get the asymptotic solution as

$$y(x) = y_0(x) + (\theta - y_0(1)) \exp\left(-\frac{a(1)(1-x)}{\varepsilon} \right), \tag{11}$$

where $y_0(x)$ is the solution of the reduced problem (obtained by setting $\varepsilon = 0$) of (10) which is given by

$$a(x)y'(x) + b(x)y(x) = F(x) \text{ with } y_0(0) = \phi(0). \tag{12}$$

Considering h small enough, the discretized form of (11) becomes

$$y(ih) = y_0(ih) + (\theta - y_0(1)) \exp\left(-\frac{a(1)(1-ih)}{\varepsilon} \right),$$

which simplifies to

$$y(ih) = y_0(ih) + (\theta - y_0(1)) \exp\left(-a(1)\left(\frac{1}{\varepsilon} - i\rho \right) \right), \tag{13}$$

where $\rho = \dfrac{h}{\varepsilon},\ h = \dfrac{1}{N}$.
To handle the effect of the perturbation parameter artificial viscosity (exponentially fitting factor $\sigma(\rho)$) is multiplied on the term containing the perturbation parameter as

$$-\varepsilon\sigma(\rho)y''(x) + a(x)y'(x) + b(x)y(x) = F(x), \tag{14}$$

with boundary conditions $y_0(0) = \phi(0)$ and $y(N) = \theta$

where $y(N)$ is evaluated by Runge-Kutta method from the reduced solution of (12).
Next, we consider the difference approximation of Eq. (9) on a uniform grid $\bar{\Omega}^N = \{x_i\}_{i=0}^N$ and denote $h = x_{i+1} - x_i$.
For any mesh function z_i, define the following difference operators

$$D^+ z_i = \frac{z_{i+1} - z_i}{h},\ D^- z_i = \frac{z_i - z_{i-1}}{h},\ D^0 z_i = \frac{z_{i+1} - z_{i-1}}{2h},\ D^+ D^- z_i = \frac{z_{i+1} - 2z_i + z_{i-1}}{h^2}, \tag{15}$$

By applying the central finite difference scheme on Eq. (14) takes the form

$$-\varepsilon\sigma(\rho)(D^+ D^- y(x_i)) + a(x_i)(D^0 y(x_i)) + b(x_i) y(x_i) = F(x_i), \tag{16}$$

with the boundary conditions $y_0(0) = \phi(0)$ and $y(N) = \theta.$
Using operator, Eq. (10) is rewritten as

$$L^h y_i = F_i \tag{17}$$

with the boundary conditions $y_0 = \phi(0)$ and $y_N = \theta.$
where

$$L^h y_i = -\varepsilon\sigma(\rho)(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}) + a(x_i)(\frac{y_{i+1} - y_{i-1}}{2h}) + b(x_i) y_i = F_i, \tag{18}$$

Multiplying Eq. (18) by h and considering h small and truncating the term $h(F_i - b(x_i) y(x_i)),$ results

$$\frac{\sigma(\rho)}{\rho}\left(y_{i-1} - 2y_i + y_{i+1}\right) + \frac{a(x_i)}{2}\left(y_{i-1} - y_{i+1}\right) = 0 \tag{19}$$

Now using Taylor's series for y_{i-1} and y_{i+1} up to first term and substituting the results in Eq. (19) into Eq. (16) and simplifying, the exponential fitting factor is obtained as

$$\sigma(\rho) = \frac{\sigma a(1)}{2}\coth\left(\frac{\sigma a(1)}{2}\right) \tag{20}$$

Assume that $\bar{\Omega}^{2N}$ denote partition of $[0,2]$ into $2N$ subintervals such that $0 = x_0 < x_1 < ... < x_N = 1$ and $1 < x_{N+1} < x_{N+2} < ... < x_{2N} = 2$ with $x_i = ih,\ h = \dfrac{2}{2N} = \dfrac{1}{N},\ i = 0,1,2,....,2N$.
Case 1: Consider Eq. (4) on the domain $\Omega_1 = (0,1)$ which is given by

$$-\varepsilon y''(x) + a(x) y'(x) + b(x) y(x) = f(x) - c(x)\phi(x-1) \tag{21}$$

Hence, the required finite difference scheme becomes

$$\left(\frac{\varepsilon\sigma(\rho)}{h^2}-\frac{a(x_i)}{2h}\right)y_{i-1}+\left(\frac{-2\varepsilon\sigma(\rho)}{h^2}+b(x_i)\right)y_i+\left(\frac{\varepsilon\sigma(\rho)}{h^2}+\frac{a(x_i)}{2h}\right)y_{i+1}=f_i-c_i\phi(x_i-N) \tag{22}$$

for $i=0,1,2,...,N.$
The numerical scheme in Eq. (22) can be written in three term recurrence relation as

$$E_i y_{i-1}+F_i y_i+G_i y_{i+1}=H_i,\qquad i=1,2,...,N \tag{23}$$

where $E_i=\frac{-\varepsilon\sigma}{h^2}-\frac{a_i}{2h},\ F_i=\frac{2\varepsilon\sigma}{h^2}+b_i,\ G_i=\frac{-\varepsilon\sigma}{h^2}+\frac{a_i}{2h},\ H_i=f_i-c_i\phi(x_i-N)$,

Case 2: Consider Eq. (4) on the domain $\Omega_2=(1,2)$, for right layer in the domain Ω_2 using exponentially fitted finite difference method, which is given by $-\varepsilon\sigma(\rho)\left(\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}\right)+a_i\left(\frac{y_i-y_{i-1}}{h}\right)+b_i y_i+c_i y(x_i-1)+\tau_1=f_i$ Similarly, this equation can be written as

$$c_i y_j+E_i y_{i-1}+F_i y_i+G_i y_{i+1}=H_i,\qquad i=N+1,N+2,...,2N-1 \tag{24}$$

where $y_j=y(x_i-1),\quad j=1,2,...,N$, $E_i=\frac{-\varepsilon\sigma}{h^2}-\frac{a_i}{2h},\ F_i=\frac{2\varepsilon\sigma}{h^2}+b_i,\ G_i=\frac{-\varepsilon\sigma}{h^2}+\frac{a_i}{2h},\ H_i=f_i$

Case 3: For $i=2N$, the composite Simpson's rule approximates the integral of $g(x)y(x)$ by

$$\int_0^2 g(x)y(x)dx=\frac{h}{3}\left(g(0)y(0)+g(2)y(2)+2\sum_{i=1}^{2N-1}g(x_{2i})y(x_{2i})+4\sum_{i=1}^{2N}g(x_{2i-1})y(x_{2i-1})\right) \tag{25}$$

Substituting Eq. (25) into Eq. (3) gives

$$y(2)-\frac{\varepsilon h}{3}\left(g(0)y(0)+g(2)y(2)+2\sum_{i=1}^{2N-1}g(x_{2i})y(x_{2i})+4\sum_{i=1}^{2N}g(x_{2i-1})y(x_{2i-1})\right)=L$$

Since $y(0)=\phi(0)$, from Eq. (2), this equation can be re-written as

$$-\frac{4\varepsilon h}{3}\sum_{i=1}^{2N}g(x_{2i-1})y(x_{2i-1})-\frac{2\varepsilon h}{3}\sum_{i=1}^{2N-1}g(x_{2i})y(x_{2i})+\left(1-\frac{\varepsilon h}{3}g(2)\right)y(2)=L+\frac{\varepsilon h}{3}g(0)y(0) \tag{26}$$

Therefore, on the whole domain $\bar{\Omega}=[0,2]$, the basic schemes to solve Eqs. (1)-(3) are the schemes given in Eq. (23), Eq. (24) and Eq. (26) together with the local truncation error of τ_1.

5. Convergence analysis

The discrete scheme corresponding to the original Eqs. (1)–(3) is as follows:

For $i = 1, 2, \ldots, N-1$,

$$L_1^N Y_i = f_i - b_i \phi_{i-N}, \tag{27}$$

For $i = N+1, \ldots, 2N-1$,

$$L_2^N Y_i = f_i, \tag{28}$$

subject to the boundary conditions:

$$Y_i = \phi_i, \; i = -N, -N+1, \ldots, 0 \tag{29}$$

$$K^N Y_{2N} = Y_{2N} - \sum_{i=1}^{2N} \frac{g_{i-1}Y_{i-1} + 4g_i Y_i + g_{i-1}Y_{i-1}}{3} h_i \tag{30}$$

And $\quad D^- Y_N = D^+ Y_N$

Where
$$L_1^N Y_i = -\varepsilon \delta^2 Y(x_i) + a(x_i) D^0 Y(x_i) + b(x_i) Y(x_i)$$
$$L_2^N Y_i = -\varepsilon \delta^2 Y(x_i) + a(x_i) D^0 Y(x_i) + b(x_i) Y(x_i) + c(x_i) Y(x_{i-N})$$

Lemma 5.1: (Discrete Maximum Principle) Assume that $\sum_{i=1}^{2N} \frac{g_{i-1} + 4g_i + g_{i-1}}{3} h_i = \rho < 1$ and mesh function $\psi(x_i)$ satisfies $\psi(x_0) \geq 0$, and $K^N \psi(x_{2N}) \geq 0$, Then $L_1^N \psi(x_i) \geq 0, \; \forall x_i \in \Omega_1^{2N}, L_2^N \psi(x_i) \geq 0, \; \forall x_i \in \Omega_2^{2N}$, and $D^+(\psi(x_N)) - D^-(\psi(x_N)) \leq 0$ imply that $\psi(x_i) \geq 0, \; \forall x_i \in \bar{\Omega}^{2N}$.

Proof: Define

$$s(x_i) = \begin{cases} \dfrac{1}{8} + \dfrac{x_i}{2}, & x_i \in [0,1] \cap \bar{\Omega}^{2N}, \\ \dfrac{3}{8} + \dfrac{x_i}{4}, & x_i \in [1,2] \cap \bar{\Omega}^{2N}, \end{cases}$$

Note that $s(x_i) > 0, \; \forall x_i \in \bar{\Omega}^{2N}$, $Ls(x_i) > 0, \; \forall x_i \in \bar{\Omega}_1^{2N} \cup \bar{\Omega}_1^{2N}, s(0) > 0, \; Ks(x_{2N}) > 0$ and $[s'](x_N) < 0$.

Let $\mu = \max\left\{ \dfrac{-\psi(x_i)}{s(x_i)} : \; x_i \in \bar{\Omega}^{2N} \right\}$. Then there exists $x_k \in \bar{\Omega}^{2N}$ such that $\psi(x_k) + \mu s(x_k) = 0$ and $\psi(x_i) + \mu s(x_i) \geq 0, \; \forall x_i \in \bar{\Omega}^{2N}$. Therefore, the function $(\psi + \mu s)$ attains its minimum at $x = x_k$. Suppose the theorem does not hold true, then $\mu > 0$.

Case (i): $x_k = x_0$

$$0 < (\psi + \mu s)(x_0) = 0, \quad \text{It is a contradiction.}$$

Case (ii): $x_k \in \Omega_1^{2N}$

$$0 < L_1^N(\psi + \mu s)(x_k) \le 0, \text{ It is a contradiction.}$$

Case (iii): $x_k = x_N$

$$0 \le [D(\psi + \mu s)'](x_N) < 0, \text{ It is a contradiction.}$$

Case (iv): $x_k \in \Omega_2^{2N}$

$$0 < L_2^N(\psi + \mu s)(x_k) \le 0, \text{ It is a contradiction.}$$

Case (v): $x_k = x_{2N}$

$$0 < K^N(\psi + \mu s)x_{2N} = (\psi + \mu s)x_{2N} - \sum_{i=1}^{2N} \frac{g_{i-1}(\psi + \mu s)x_{i-1} + 4g_i(\psi + \mu s)x_i + g_{i-1}(\psi + \mu s)x_{i-1}}{3} h_i \le 0$$

It is a contradiction. Hence the proof of the theorem.

Lemma 5.2: Let $\psi(x_i)$ be any mesh function then for $0 \le i \le 2N$,

$$|\psi(x_i)| \le C \max\left\{|\psi(x_0)|, |K^N\psi(x_{2N})|, \max_{i\in\Omega_1^{2N}\cup\Omega_2^{2N}} |L^N\psi(x_i)|\right\}$$

Proof: For the proof refer [16]

The following theorem shows the parameter uniform convergence of the scheme developed.

Theorem 5.1: Let $y(x_i)$ and y_i be respectively the exact solution of Eqs. (1)-(3) and numerical solutions of Eq. (17). Then for sufficiently large N, the following parameter uniform error estimate holds:

$$\sup_{0<\varepsilon\le 1} \|y(x_i) - y_i\| \le CN^{-2} \tag{31}$$

Proof: Let us consider the local truncation error defined as

$$L^h(y(x_i) - y_i) = -\varepsilon\sigma(\rho)\left(\frac{d^2}{dx^2} - D^+D^-\right)y(x_i) + a(x_i)\left(\frac{d}{dx} - D^0\right)y(x_i) \tag{32}$$

where $\varepsilon\sigma(\rho) = a(1)\dfrac{N^{-1}}{2}\coth\left(a(1)\dfrac{N^{-1}}{2\varepsilon}\right)$ since $\dfrac{N^{-1}}{2\varepsilon}$. In our assumption $\varepsilon \le h = N^{-1}$.

By considering is fixed and taking the limit for $\varepsilon \to 0$, we obtain the following

$$\lim_{\varepsilon\to 0} \varepsilon\sigma(\rho) = \lim_{\varepsilon\to 0} a(1)\frac{N^{-1}}{2}\coth\left(a(1)\frac{N^{-1}}{2\varepsilon}\right) = CN^{-1}$$

From Taylor series expansion, the bound for the difference becomes

$$\begin{cases} \left\|\left(\dfrac{d^2}{dx^2} - D^+D^-\right)y(x_i)\right\| \le CN^{-3}\left\|\dfrac{d^4(y(x_i))}{dx^4}\right\| \\ \left\|\left(\dfrac{d}{dx} - D^0\right)y(x_i)\right\| \le CN^{-2}\left\|\dfrac{d^3(y(x_i))}{dx^3}\right\| \end{cases}$$

where $\left\|\dfrac{d^k(y(x_i))}{dx^k}\right\| = \sup_{x_i\in(x_0,x_N)}\left(\dfrac{d^k y(x_i)}{dx^k}\right)$, $k = 3, 4$.

Now using the bounds and the assumption $\varepsilon \le N^{-1}$, (32) reduces to

$$\begin{aligned}\left\|L^h(y(x_i)-y_i)\right\| &= \left\|-\varepsilon\sigma(\rho)\left(\frac{d^2}{dx^2}-D^+D^-\right)y(x_i)+a(x_i)\left(\frac{d}{dx}-D^0\right)y(x_i)\right\| \\ &\le \left\|-\varepsilon\sigma(\rho)\left(\frac{d^2}{dx^2}-D^+D^-\right)y(x_i)\right\|+\left\|a(x_i)\left(\frac{d}{dx}-D^0\right)y(x_i)\right\| \\ &\le CN^{-3}\left\|\frac{d^4(y(x_i))}{dx^4}\right\|+CN^{-2}\left\|\frac{d^3(y(x_i))}{dx^3}\right\|\end{aligned} \tag{33}$$

Here, the target is to show the scheme convergence independent on the number of mesh points. By using the bounds for the derivatives of the solution in Lemma 3.4, we obtain

$$\begin{aligned}\left\|L^h(y(x_i)-y_i)\right\| &\le CN^{-3}\left\|\frac{d^4(y(x_i))}{dx^4}\right\|+CN^{-2}\left\|\frac{d^3(y(x_i))}{dx^3}\right\| \\ &\le CN^{-3}\left(1+\varepsilon^{-4}\exp\left(\frac{-\alpha(1-x_j)}{\varepsilon}\right)\right)+CN^{-2}\left(1+\varepsilon^{-3}\exp\left(\frac{-\alpha(1-x_j)}{\varepsilon}\right)\right) \\ &\le CN^{-2}\left(1+\varepsilon^{-4}\exp\left(\frac{-\alpha(1-x_j)}{\varepsilon}\right)\right),\quad \text{since } \varepsilon^{-4}\ge\varepsilon^{-3}\end{aligned} \tag{34}$$

Lemma 5.3: For a fixed mesh and for $\varepsilon \to 0$, it holds

$$\lim_{\varepsilon\to 0}\max_{1\le j\le N-1}\frac{\exp\left(\dfrac{-\alpha(1-x_j)}{\varepsilon}\right)}{\varepsilon^m}=0,\quad m=1,2,3,\ldots \tag{35}$$

Proof: Refer from [19]
By using Lemma 5.3 into Eq. (34), results to

$$\left\|L^h(y(x_i)-y_i)\right\|\le CN^{-2} \tag{36}$$

Hence, by discrete maximum principle, we obtain

$$\left\|y(x_i)-y_i\right\|\le CN^{-2} \tag{37}$$

Thus, result of Eq. (37) shows Eq. (31). Hence the proof.
Remark: A similar analysis for convergence may be carried out for finite difference scheme (24).

6. Numerical Examples and Results

In this section, four examples are given to illustrate the numerical method discussed above. The exact solutions of the test problems are not known. Therefore, we use the double mesh principle to estimate the error and compute the experiment rate of convergence to the computed solution. For this we put $E_\varepsilon^h = \max_{0\le i\le 2N}\left|Y_i^N - Y_{2i}^{2N}\right|$ where Y_i^N and Y_{2i}^{2N} are the i^{th} components of the numerical solutions on meshs of N and $2N$, respectively. We compute the uniform error and the rate of convergence as $E^h = \max_\varepsilon E_\varepsilon^h$ and $R^h = \log_2\left(\frac{E^N}{E^{2N}}\right)$. The numerical results are presented for the values of the perturbation parameter $\varepsilon \in \{10^{-4}, 10^{-8}, ..., 10^{-20}\}$.

Example 1:

$$
\begin{aligned}
&-\varepsilon y''(x) + 3y'(x) - y(x-1) = 0, \quad x \in (0,1)\cup(1,2)\\
&y(x) = 1, \quad x \in [-1,0]\\
&y(2) - \varepsilon\int_0^2 \frac{x}{3} y(x)dx = 2
\end{aligned}
$$

Table 1. Maximum absolute errors and rate of convergence for Example 1 at number of mesh points $2N$

ε	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
10^{-4}	6.8902e-03	3.4587e-03	1.7327e-03	8.6721e-04	4.3382e-04
	0.9943	0.9972	0.9986	0.9993	
10^{-8}	6.8902e-03	3.4587e-03	1.7327e-03	8.6721e-04	4.3382e-04
	0.9943	0.9972	0.9986	0.9993	
10^{-12}	6.8902e-03	3.4587e-03	1.7327e-03	8.6721e-04	4.3382e-04
	0.9943	0.9972	0.9986	0.9993	
10^{-16}	6.8902e-03	3.4587e-03	1.7327e-03	8.6721e-04	4.3382e-04
	0.9943	0.9972	0.9986	0.9993	
10^{-20}	6.8902e-03	3.4587e-03	1.7327e-03	8.6721e-04	4.3382e-04
	0.9943	0.9972	0.9986	0.9993	

Table 2. Comparisons of maximum absolute errors and rate of convergence for Example 1 at number of mesh points $2N$

$\varepsilon\downarrow N\rightarrow$	32	64	128	256	512
Present Method					
2^{-10}	6.8902e-03	3.4587e-03	1.7327e-03	8.6773e-04	4.3760e-04
2^{-11}	6.8902e-03	3.4587e-03	1.7327e-03	8.6721e-04	4.3382e-04
2^{-12}	6.8902e-03	3.4587e-03	1.7327e-03	8.6721e-04	4.3382e-04
2^{-13}	6.8902e-03	3.4587e-03	1.7327e-03	8.6721e-04	4.3382e-04
2^{-14}	6.8902e-03	3.4587e-03	1.7327e-03	8.6721e-04	4.3382e-04
2^{-15}	6.8902e-03	3.4587e-03	1.7327e-03	8.6721e-04	4.3382e-04
2^{-16}	6.8902e-03	3.4587e-03	1.7327e-03	8.6721e-04	4.3382e-04
2^{-17}	6.8902e-03	3.4587e-03	1.7327e-03	8.6721e-04	4.3382e-04
2^{-18}	6.8902e-03	3.4587e-03	1.7327e-03	8.6721e-04	4.3382e-04
2^{-19}	6.8902e-03	3.4587e-03	1.7327e-03	8.6721e-04	4.3382e-04
2^{-20}	6.8902e-03	3.4587e-03	1.7327e-03	8.6721e-04	4.3382e-04
E^h	**6.8902e-03**	**3.4587e-03**	**1.7327e-03**	**8.6721e-04**	**4.3382e-04**
R^h	**0.9943**	**0.9972**	**0.9986**	**0.9993**	
Result in [16]					
2^{-10}	8.7402e−03	4.0726e−03	1.9156e−03	9.1036e−04	4.3976e−04
2^{-11}	9.3259e-03	4.4288e-03	2.1232e-03	1.0265e-03	5.0201e-04
2^{-12}	9.7404e-03	4.6808e-03	2.2700e-03	1.1087e-03	5.4603e-04
2^{-13}	1.0033e-02	4.8591e-03	2.3738e-03	1.1668e-03	5.7716e-04
2^{-14}	1.0241e-02	4.9852e-03	2.4472e-03	1.2079e-03	5.9917e-04
2^{-15}	1.0387e-02	5.0744e-03	2.4991e-03	1.2370e-03	6.1474e-04
2^{-16}	1.0491e-02	5.1374e-03	2.5358e-03	1.2575e-03	6.2574e-04
2^{-17}	1.0564e-02	5.1820e-03	2.5618e-03	1.2720e-03	6.3352e-04
2^{-18}	1.0616e-02	5.2135e-03	2.5801e-03	1.2823e-03	6.3902e-04
2^{-19}	1.0653e-02	5.2358e-03	2.5931e-03	1.2896e-03	6.4292e-04
2^{-20}	1.0679e-02	5.2516e-03	2.6023e-03	1.2947e-03	6.4567e-04
E^h	**1.0679e-02**	**5.2516e-03**	**2.6023e-03**	**1.2947e-03**	**6.4567e-04**
R^h	**1.0240**	**1.0129**	**1.0071**	**1.0038**	

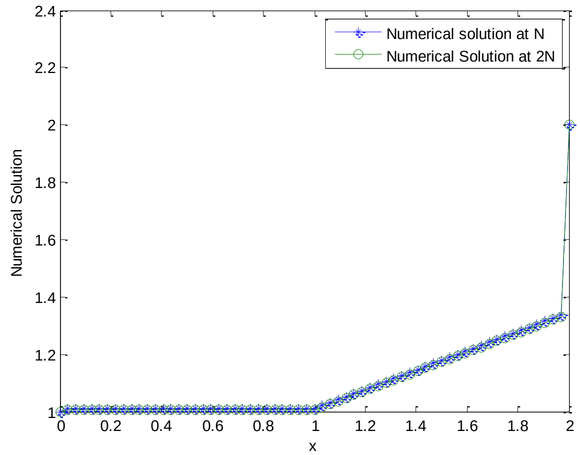

Fig. 1. The behavior of the Numerical Solution for Example 1 at $\varepsilon = 10^{-12}$ and $N = 32$.

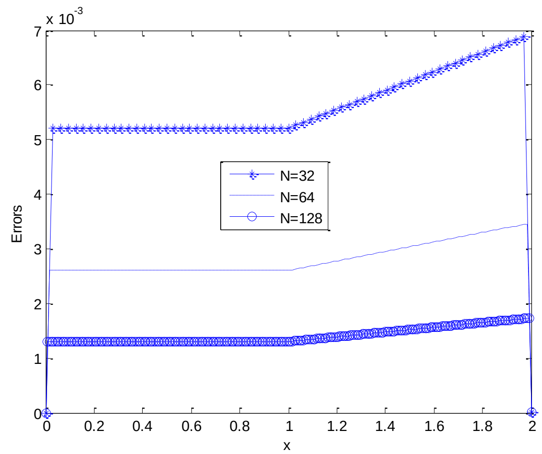

Fig. 2. Point wise absolute error of Example 1 at $\varepsilon = 10^{-12}$ with different mesh size h.

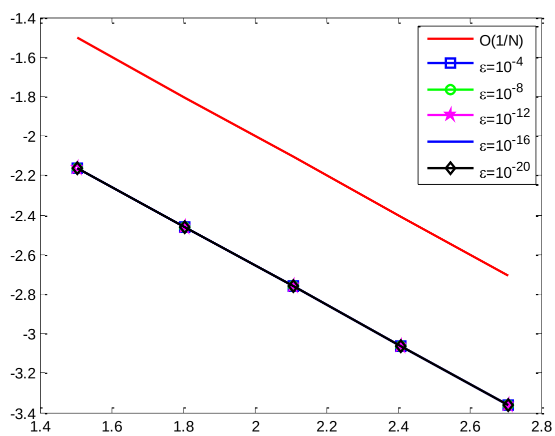

Fig. 3. ε-uniform convergence with NSFDM in Log-Log scale for example 1

Example 2:

$$-\varepsilon y''(x)+(x+10)y'(x)-y(x-1)=x^2,\quad x\in(0,1)\cup(1,2)$$
$$y(x)=1,\quad x\in[-1,0]$$
$$y(2)-\varepsilon\int_0^2 \frac{x}{3}y(x)dx=2$$

Table 3. Maximum absolute errors and rate of convergence for Example 2 at number of mesh points $2N$

ε	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
10^{-4}	1.5579e-03	7.7976e-04	3.9021e-04	1.9520e-04	9.7628e-05
	0.9985	0.9988	0.9993	0.9996	
10^{-8}	1.5579e-03	1.5579e-03	3.9021e-04	1.9520e-04	9.7628e-05
	0.9985	0.9988	0.9993	0.9996	
10^{-12}	1.5579e-03	1.5579e-03	3.9021e-04	1.9520e-04	9.7628e-05
	0.9985	0.9988	0.9993	0.9996	
10^{-16}	1.5579e-03	1.5579e-03	3.9021e-04	1.9520e-04	9.7628e-05
	0.9985	0.9988	0.9993	0.9996	
10^{-20}	1.5579e-03	1.5579e-03	3.9021e-04	1.9520e-04	9.7628e-05
	0.9985	0.9988	0.9993	0.9996	

Table 4. Comparison of Maximum absolute errors and rate of convergence for Example 2 at number of mesh points $2N$

$\varepsilon\downarrow N\rightarrow$	32	64	128	256	512
Present Method					
2^{-10}	1.5579e-03	7.7976e-04	3.9021e-04	1.9520e-04	9.7633e-05
2^{-11}	1.5579e-03	7.7976e-04	3.9021e-04	1.9520e-04	9.7633e-05
2^{-12}	1.5579e-03	7.7976e-04	3.9021e-04	1.9520e-04	9.7633e-05
2^{-13}	1.5579e-03	7.7976e-04	3.9021e-04	1.9520e-04	9.7633e-05
2^{-14}	1.5579e-03	7.7976e-04	3.9021e-04	1.9520e-04	9.7633e-05
2^{-15}	1.5579e-03	7.7976e-04	3.9021e-04	1.9520e-04	9.7633e-05
2^{-16}	1.5579e-03	7.7976e-04	3.9021e-04	1.9520e-04	9.7628e-05
2^{-17}	1.5579e-03	7.7976e-04	3.9021e-04	1.9520e-04	9.7628e-05
2^{-18}	1.5579e-03	7.7976e-04	3.9021e-04	1.9520e-04	9.7628e-05
2^{-19}	1.5579e-03	7.7976e-04	3.9021e-04	1.9520e-04	9.7628e-05
2^{-20}	1.5579e-03	7.7976e-04	3.9021e-04	1.9520e-04	9.7628e-05
E^h	**1.5579e-03**	**7.7976e-04**	**3.9021e-04**	**1.9520e-04**	**9.7628e-05**
R^h	**0.9985**	**0.9988**	**0.9993**	**0.9993**	
Result in [16]					
2^{-10}	5.5958e-03	2.5488e-03	1.1977e-03	5.7296e-04	2.7750e-04
2^{-11}	5.8142e-03	2.6717e-03	1.2671e-03	6.1165e-04	2.9860e-04
2^{-12}	5.9692e-03	2.7588e-03	1.3161e-03	6.3898e-04	3.1349e-04
2^{-13}	6.0790e-03	2.8204e-03	1.3508e-03	6.5829e-04	3.2400e-04

2^{-14}	6.1568e-03	2.8640e-03	1.3753e-03	6.7194e-04	3.3142e-04
2^{-15}	6.2119e-03	2.8948e-03	1.3927e-03	6.8158e-04	3.3667e-04
2^{-16}	6.2509e-03	2.9166e-03	1.4049e-03	6.8840e-04	3.4038e-04
2^{-17}	6.2785e-03	2.9321e-03	1.4136e-03	6.9322e-04	3.4300e-04
2^{-18}	6.2980e-03	2.9430e-03	1.4197e-03	6.9662e-04	3.4485e-04
2^{-19}	6.3118e-03	2.9507e-03	1.4241e-03	6.9903e-04	3.4616e-04
2^{-20}	6.3216e-03	2.9562e-03	1.4271e-03	7.0074e-04	3.4709e-04
E^h	**6.3216e-03**	**2.9562e-03**	**1.4271e-03**	**7.0074e-04**	**3.4709e-04**
R^h	**1.0965**	**1.0505**	**1.0262**	**1.0135**	

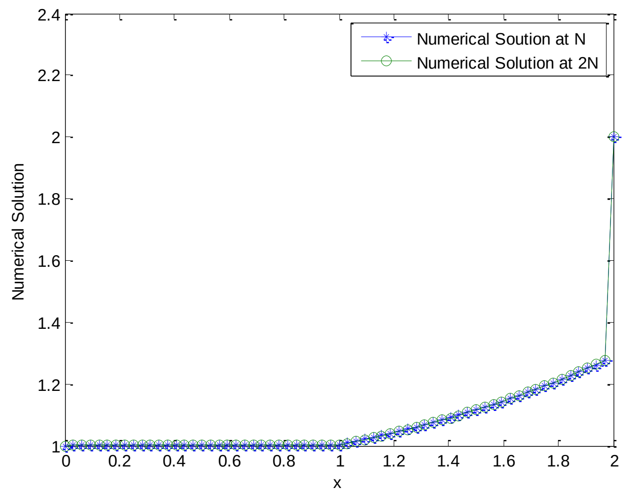

Fig. 4. The behavior of numerical solution for example 2 at $\varepsilon = 10^{-12}$ and $N = 32$.

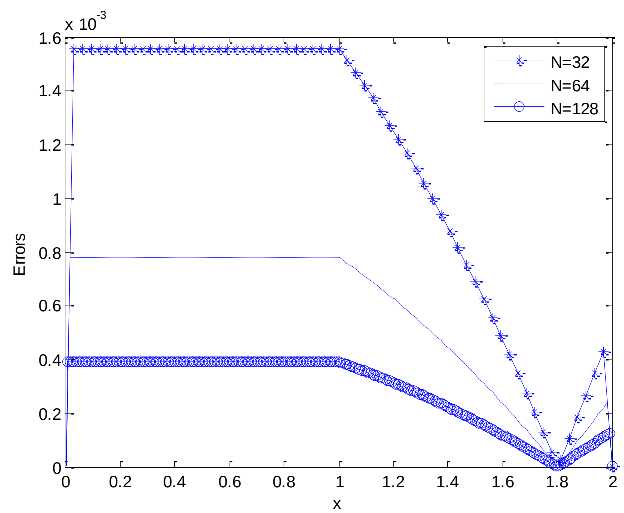

Fig. 5. Point wise absolute error of Example 2 at $\varepsilon = 10^{-12}$ with different mesh size h.

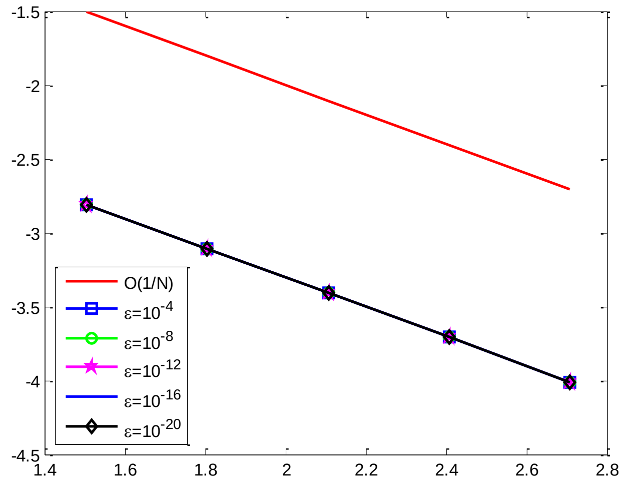

Fig. 6. ε-uniform convergence with NSFDM in Log-Log scale for example 2

7. Discussion and Conclusion

This study introduces fitted operator finite difference numerical method (NSFDM) for solving singularly perturbed delay differential equations with integral boundary condition. The behavior of the continuous solution of the problem is studied and shown that it satisfies the continuous stability estimate and the derivatives of the solution are also bounded. The numerical scheme is developed on uniform mesh using exponential fitted operator in the given differential equation. The integral boundary condition is treated using Simpson's rule. The stability of the developed numerical method is established and its uniform convergence is proved. To validate the applicability of the method, two model problems are considered for numerical experimentation for different values of the perturbation parameter and mesh points. The numerical results are tabulated in terms of maximum absolute errors, numerical rate of convergence and uniform errors (see Tables 1-4). Further, behavior of the numerical solution (Figure 1 and 4), point-wise absolute errors (Figure 2 and 5) and the ε -uniform convergence of the method is shown by the log-log plot (Figure 3 and 6). The method is shown to be ε-uniformly convergent with order of convergence $O(h)$. The performance of the proposed scheme is investigated by comparing with prior study (see Table 2 and 4). The proposed method gives more accurate, stable and ε-uniform numerical result.

Reference

[1] Derstine M. W., Gibbs F.A.H.H.M. and Kaplan D. L., Bifurcation gap in a hybrid optical system, Phys. Rev. A, 26, 3720-3722, 1982.

[2] Longtin A. and J. Milton J., Complex oscillations in the human pupil light reflex with mixed and delayed feedback, Math. Biosci, 90, 183-199, 1988.

[3] Mackey M.C. and Glass L., Oscillation and chaos in physiological control systems, Science. 197, 287-289, 1977.

[4] Mallet-Paret J. and Nussbaum R. D., A differential-delay equations arising in optics and physiology, SIAMJ. Math. Anal. 20, 249-292, 1989.

[5] Lange C.G. and Miura R.M., Singular perturbation analysis of boundary value problems for differential difference equations, SIAM J. Appl. Math. Vol. 42(3), 502-531, 1982.

[6] Kadalbajoo M.K and Sharma K.K., An exponentially fitted finite difference scheme for solving boundary value problems for singularly-perturbed differential-difference equations: small shifts of mixed type with layer behaviour, J. Comput. Anal. Appl. Vol. 8(2), 151-171, 2006.

[7] Patidar K. C. and Sharma K.K., ε-uniformly convergent non-standard finite difference methods for singularly perturbed differential difference equations with small delay, Appl. Math. Comput. Vol. 175(1), 864-890, 2006.

[8] Kadalbajoo M.K. and Sharma K.K., An ε-uniform convergent method for a general boundary-value problem for singularly perturbed differential-difference equations: Small shifts of mixed type with layer behaviour, J. Comput. Methods Sci. Eng, 6, 39-55, 2006.

[9] Kadalbajoo M.K. and Sharma K.K., Numerical treatment of boundary value problems for second order singularly perturbed delay differential equations, Comput. Appl. Math, vol. 24(2), 151-172, 2005.

[10] Kadalbajoo M.K. and Sharma K.K., Numerical treatment for singularly perturbed nonlinear differential difference equations with negative shift, Nonlinear Anal. Theory Methods Appl, vol. 63(5), e1909-e1909, 2005.

[11] Kadalbajoo M.K. and Sharma K.K., Numerical analysis of singularly perturbed delay differential equations with layer behaviour, Appl. Math. Comput, vol. 157(1), 11-28, 2004.

[12] Kadalbajoo M.K. and Sharma K.K., Numerical analysis of boundary-value problems for singularly perturbed differential-difference equations with small shifts of mixed type, J. Optim. Theory Appl, vol.115(1), 145-163, 2002.

[13] Hongijiong T., Numerical methods for singularly perturbed delay differential equations, In proceeding of the International conference on Boundary and Interior Layers, Computational and Asymptotic Methods-BAIL 2004, ONERA, Toulouse.

[14] Amiraliyev G. M and Cimen E., Numerical method for a singularly perturbed convection-diffusion problem with delay, Appl. Math. Comput. 216, 2351-2359, 2010.

[15] Subburayan V. and Ramanujam N., An initial value technique for singularly perturbed convection-diffusion problems with a negative shift, J. Optim Theory Appl. 158, 234-250, 1992.

[16] Sakar E. and Tamilselvan A., Singularly perturbed delay differential equations of convection–diffusion type with integral boundary condition, J. Appl. Math. Comput, 2018, https://doi.org/10.1007/s12190-018-1198-4.

[17] Clavero C., Gracia J.L. and Jorge J.C., High-order numerical methods for one dimensional parabolic singularly perturbed problems with regular layers, Numer. Methods Partial differential equations, vol. 21(1), 149-169, 2005.

[18] OMalley R.E., Singular Perturbation Methods for Ordinary Differential Equations. Springer-Verlag,New York, (1991).

[19] Woldaregay M.M. and Duressa G.F., parameter uniform numerical method for singularly perturbed differential difference equations, J. Nigerian Math. Soc. Vol.38(2), 223-245, 2009.