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Tenth Order Compact Finite Difference Method for Solving Singularly Perturbed 1D Reaction - Diffusion Equations

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Abstract

In this paper, tenth order compact finite difference method have been presented for solving singularly perturbed two-point boundary value problems of 1D reaction-diffusion equations. The derivatives in the given differential equation have been replaced by finite difference approximations and transformed to tri-diagonal system which can easily be solved by Discrete Invariant Imbedding algorithm. The theoretical error bounds have been established for the method. Three model examples have been considered to check the applicability of the proposed method. The numerical results presented in tables show that the present method approximates the exact solution very well.

Keywords: Singular perturbation, compact differential difference method, reaction-diffusion equations.

1. Introduction

Any differential equation in which the highest order derivative is multiplied by a small positive parameter $\varepsilon\left(0<\varepsilon\ll 1\right)$ is called singular perturbation problem and the parameter is known as the perturbation parameter. These types of problems arise frequently in many fields of applied mathematics and engineering, like quantum mathematics, fluid dynamics, chemical reactions, electrical network, nuclear physics, elasticity, hydro-dynamics, modeling of semiconductor devices, diffraction theory and reaction-diffusion processes and many other allied areas. Classical computational approaches to singularly perturbed problems are known to be inadequate as they require extremely large numbers of mesh points to produce satisfactory computed solutions Farrell et al. [1] and Roos et al. [2]. Detailed discussions on the theory of asymptotical and numerical solutions of singular perturbation problems have been published (see [3 - 9]). So, the treatment of singularly perturbed problems presents severe difficulties that have to be addressed to ensure accurate numerical solutions (see [10 - 13]).

It is well-known fact that the solution of singular perturbation problem exhibits a multi-scale character that is; there are thin layer(s) where the solution varies rapidly, while away from the layer(s) the solution behaves regularly and varies slowly. However, most of the existing classical finite difference methods which have been used in solving singular perturbation problems give good result only when the mesh size is much less than the perturbation parameter which is very costly and time consuming.

In this paper, tenth order compact finite difference method is presented for solving second-order self-adjoint singularly perturbed 1D reaction-diffusion problems. Compact finite difference method is a finite difference method which employs a linear combination of three consecutive points of derivatives to approximate a linear combination of the same three consecutive values of a function $y(x_j)$, $j = i-1, i, i+1$. To validate the efficiency of the method, three modal examples are solved for different values of the parameter ε, mesh length h and compare the maximum absolute error with the more currently published papers.

2. Description of the Method

Consider the following singularly perturbed 1D reaction-diffusion equation of the form:

$$-\varepsilon y''(x) + a(x)y(x) = f(x);\ 0 \le x \le 1, \tag{1}$$

with the Dirichlet boundary conditions

$$y(0) = \alpha,\ y(1) = \beta \tag{2}$$

where ε is a small positive parameter (diffusion coefficient) such that $0 < \varepsilon << 1$ and α, β are given constants and $a(x)$; $f(x)$ are assumed to be sufficiently continuously differentiable functions such that $a(x) \ge \gamma > 0$ for every $x \in [0, 1]$, where γ is some positive constant.

To describe the scheme, we divide the interval $[0, 1]$ into N equal subintervals of mesh length h Let $0 = x_1, x_2, ..., x_N = 1$ be the mesh points. Then, we have $x_i = x_0 + ih$, $i = 0,1,...,N$.

For convenience, let $a(x_i) = a_i$, $f(x_i) = f_i$, $y(x_i) = y_i$, $y'(x_i) = y_i'$, $y''(x_i) = y_i''$, $y^{(n)}(x_i) = y_i^{(n)}$.

Assume that $y(x)$ has continuous higher order derivatives on $[0, 1]$.

Using Taylor series expansion, we have:

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \frac{h^3}{3!}y_i''' + \frac{h^4}{4!}y_i^{(4)} + \frac{h^5}{5!}y_i^{(5)} + \frac{h^6}{6!}y_i^{(6)} + \frac{h^7}{7!}y_i^{(7)} + \frac{h^8}{8!}y_i^{(8)} + \frac{h^9}{9!}y_i^{(9)} + \frac{h^{10}}{10!}y_i^{(10)} + \frac{h^{11}}{11!}y_i^{(11)} + \frac{h^{12}}{12!}y_i^{(12)} + O(h^{13}) \tag{3}$$

$$y_{i-1} = y_i - hy_i' + \frac{h^2}{2!}y_i'' - \frac{h^3}{3!}y_i''' + \frac{h^4}{4!}y_i^{(4)} - \frac{h^5}{5!}y_i^{(5)} + \frac{h^6}{6!}y_i^{(6)} - \frac{h^7}{7!}y_i^{(7)} + \frac{h^8}{8!}y_i^{(8)} - \frac{h^9}{9!}y_i^{(9)} + \frac{h^{10}}{10!}y_i^{(10)} - \frac{h^{11}}{11!}y_i^{(11)} + \frac{h^{12}}{12!}y_i^{(12)} + O(h^{13}) \tag{4}$$

Subtracting Eq. (4) from Eq. (3), we obtain the second order finite difference approximation $(\delta_c^1 y_i)$ for the first derivative of y_i is:

$$\delta_c^1 y_i = \frac{y_{i+1} - y_{i-1}}{2h} + \tau_1 \tag{5}$$

where $\tau_1 = -\frac{h^2}{6}y_i'''$

Adding Eqs. (3) and (4), we obtain the second order finite difference approximation $(\delta_c^2 y_i)$ for the second derivative of y_i is:

$$\delta_c^2 y_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \tau_2 \tag{6}$$

where $\tau_2 = -\frac{h^2}{12} y_i^{(4)}$

Substituting Eqs. (3) and (4) into Eqs. (5) and (6) yields

$$\delta_c^1 y_i = y_i' + \frac{h^2}{6} y_i''' + \frac{h^4}{120} y_i^{(5)} + \frac{h^6}{5{,}040} y_i^{(7)} + \frac{h^8}{362{,}880} y_i^{(9)} + \tau_3 \tag{7}$$

$$\text{where}\, \tau_3 = \frac{h^{10}}{39{,}916{,}800} y_i^{(11)} + \tau_1$$

$$\delta_c^2 y_i = y_i'' + \frac{h^2}{12} y_i^{(4)} + \frac{h^4}{360} y_i^{(6)} + \frac{h^6}{20{,}160} y_i^{(8)} + \frac{h^8}{1{,}814{,}400} y_i^{(10)} + \tau_4 \tag{8}$$

$$\text{where}\, \tau_4 = \frac{h^{10}}{239{,}500{,}800} y_i^{(12)} - \frac{h^2}{12} y_i^{(4)}$$

Writing Eq. (1) at discretized mesh, we obtain

$$y_i'' = \frac{a_i}{\varepsilon} y - \frac{f_i}{\varepsilon} \tag{9}$$

Differentiating Eq. (9) twice, four and six times respectively, we have

$$y_i^{(4)} = \frac{a_i}{\varepsilon} y_i'' - \frac{f_i''}{\varepsilon} \tag{10}$$

$$y_i^{(6)} = \frac{a_i^2}{\varepsilon^2} y_i'' - \frac{a_i}{\varepsilon^2} f_i'' - \frac{f_i^{(4)}}{\varepsilon} \tag{11}$$

$$y_i^{(8)} = \frac{a_i^3}{\varepsilon^3} y_i'' - \frac{a_i^2}{\varepsilon^3} f_i'' - \frac{a_i}{\varepsilon^2} f_i^{(4)} - \frac{f_i^{(6)}}{\varepsilon} \tag{12}$$

By applying δ_c^2 to $y_i^{(8)}$ in Eq. (6), we obtain:

$$y_i^{(10)} = \delta_c^2 y_i^{(8)} - T_2^{(8)} \tag{13}$$

Substituting Eq. (13) in Eq. (6) yields:

$$\delta_c^2 y_i = y_i'' + \frac{h^2}{12} y_i^{(4)} + \frac{h^4}{360} y_i^{(6)} + \left[\frac{h^6}{20{,}160} + \frac{h^8}{1{,}814{,}400} \delta_c^2 \right] y_i^{(8)} + T_5 \tag{14}$$

$$\text{where}\, \tau_5 = \tau_4 - \frac{h^8}{1{,}814{,}400} \tau_2^{(8)} = \frac{h^{10}}{19{,}958{,}400} y_i^{(12)} - \frac{h^2}{12} y_i^{(4)}$$

Substituting Eqs. (10), (11) and (12) into Eq. (14), gives:

$$y_i'' = \frac{\delta_c^2 y_i + \left(\frac{h^2}{12\varepsilon} + \frac{a_i h^4}{360\varepsilon^2} + \frac{a_i^2 h^6}{20{,}160\varepsilon^3} + \frac{a_i^2 h^8}{1{,}814{,}400\varepsilon^3} \delta_c^2 \right) f_i'' + \left(\frac{h^4}{360\varepsilon} + \frac{a_i h^6}{20{,}160\varepsilon^2} + \frac{a_i h^8}{1{,}814{,}400\varepsilon^2} \delta_c^2 \right) f_i^{(4)} + \left(\frac{h^6}{20{,}160\varepsilon} + \frac{h^8}{1{,}814{,}400\varepsilon} \delta_c^2 \right) f_i^{(6)} - \tau_5}{\left(1 + \frac{a_i h^2}{12\varepsilon} + \frac{a_i^2 h^4}{360\varepsilon^2} + \frac{a_i^3 h^6}{20{,}160\varepsilon^3} + \frac{a_i^3 h^8}{1{,}814{,}400\varepsilon^3} \delta_c^2 \right)} \tag{15}$$

By substituting Eq. (15) for the value y'' into Eq. (1) and rearranging, we obtain:

$$\left(-\varepsilon+\frac{a_i^4h^8}{1{,}814{,}400\varepsilon^3}\right)\delta_c^2y_i+\left(a_i+\frac{a_i^2h^2}{12\varepsilon}+\frac{a_i^3h^4}{360\varepsilon^2}+\frac{a_i^4h^6}{20{,}160\varepsilon^3}\right)y_i$$
$$=\left(1+\frac{a_ih^2}{12\varepsilon}+\frac{a_i^2h^4}{360\varepsilon^2}+\frac{a_i^3h^6}{20{,}160\varepsilon^3}\right)f_i\ \frac{a_i^3h^8}{1{,}814{,}400\varepsilon^3}\delta_c^2f_i+$$
$$\left(\frac{h^2}{12}+\frac{a_ih^4}{360\varepsilon}+\frac{a_i^2h^6}{20{,}160\varepsilon^2}\right)f_i''+\frac{a_i^2h^8}{1{,}814{,}400\varepsilon^2}\delta_c^2f_i''+\left(\frac{h^4}{360}+\frac{a_ih^6}{20{,}160\varepsilon}\right)f_i^{(4)}+ \tag{16}$$
$$\frac{a_ih^8}{1{,}814{,}400\varepsilon}\delta_c^2f_i^{(4)}+\frac{h^6}{20{,}160}f_i^{(6)}+\frac{h^8}{1{,}814{,}400}\delta_c^2f_i^{(6)}-\varepsilon\tau_5$$

Substituting Eq. (6) into Eq. (16) for $(\delta_c^2y_i)$ together with

$$\delta_c^2f_i=\frac{f_{i+1}-2f_i+f_{i-1}}{h^2},\qquad \delta_c^2f_i''=\frac{f_{i+1}''-2f_i''+f_{i-1}''}{h^2},$$
$$\delta_c^2f_i^{(4)}=\frac{f_{i+1}^{(4)}-2f_i^{(4)}+f_{i-1}^{(4)}}{h^2}\qquad \delta_c^2f_i^{(6)}=\frac{f_{i+1}^{(6)}-2f_i^{(6)}+f_{i-1}^{(6)}}{h^2}$$

and rearranging, we obtain the equivalent three-term recurrence relation given by:

$$-E_iy_{i-1}+F_iy_i-G_iy_{i+1}=H_i,\qquad i=1,2,....,N-1 \tag{17}$$

where $E_i=\frac{\varepsilon}{h^2}-\frac{a_i^4h^6}{1{,}814{,}400\varepsilon^3}=G_i$

$$F_i=\frac{2\varepsilon}{h^2}+a_i+\frac{a_i^2h^2}{12\varepsilon}+\frac{a_i^3h^4}{360\varepsilon^2}+\frac{11a_i^4h^6}{226{,}800\varepsilon^3},$$
$$H_i=\frac{a_i^3h^6}{1{,}814{,}400\varepsilon^3}f_{i-1}+\left(1+\frac{a_ih^2}{12\varepsilon}+\frac{a_i^2h^4}{360\varepsilon^2}+\frac{11a_i^3h^6}{226{,}800\varepsilon^3}\right)f_i+\frac{a_i^3h^6}{1{,}814{,}400\varepsilon^3}f_{i+1}+$$
$$\frac{a_i^2h^6}{1{,}814{,}400\varepsilon^2}f_{i-1}''+\left(\frac{h^2}{12}+\frac{a_ih^4}{360\varepsilon}+\frac{11a_i^2h^6}{226{,}800\varepsilon^2}\right)f_i''+\frac{a_i^2h^6}{1{,}814{,}400\varepsilon^2}f_{i+1}''+$$
$$\frac{a_ih^6}{1{,}814{,}400\varepsilon}f_{i-1}^{(4)}+\left(\frac{h^4}{360}+\frac{11a_ih^6}{226{,}800\varepsilon}\right)f_i^{(4)}+\frac{a_ih^6}{1{,}814{,}400\varepsilon}f_{i+1}^{(4)}+$$
$$\frac{h^6}{1{,}814{,}400}f_{i-1}^{(6)}+\frac{11h^6}{226{,}800}f_i^{(6)}+\frac{h^6}{1{,}814{,}400}f_{i+1}^{(6)}$$

Eq. (17) gives us the tri-diagonal system which can easily be solved by applying Thomas Algorithm.

3. Convergence Analysis

Writing the tri-diagonal system Eq. (17) above in matrix vector form, we obtain

$$AY = C \tag{18}$$

where $A = (m_{ij}), 1 \le i,\ j \le N-1$ is a tri-diagonal matrix of order N, with

$$m_{ii+1} = -\frac{\varepsilon}{h^2} + \frac{a_i^4 h^6}{1{,}814{,}400\varepsilon^3}$$

$$m_{ii} = \frac{2\varepsilon}{h^2} + a_i + \frac{a_i^2 h^2}{12\varepsilon} + \frac{a_i^3 h^4}{360\varepsilon^2} + \frac{11 a_i^4 h^6}{226{,}800\varepsilon^3}$$

$$m_{ii-1} = -\frac{\varepsilon}{h^2} + \frac{a_i^4 h^6}{1{,}814{,}400\varepsilon^3}$$

and $C = H_i$

For $i = 1,2,3,..., N-1$ and with the local truncation error

$$\tau_i(h_i) = \frac{a_i^4 h^{10}}{21{,}772{,}800\varepsilon^3} y_i^{(4)} - \frac{\varepsilon h^{10}}{19{,}958{,}400} y_i^{(12)} \tag{19}$$

We also have

$$A\bar{Y} - \tau(h) = C \tag{20}$$

where $\bar{Y} = (\bar{y}_0, \bar{y}_1, \bar{y}_2, ..., \bar{y}_N)^t$ denotes the exact solution and

$\tau(h) = (\tau_1(h_0), \tau_2(h_1), ..., \tau_N(h_N))^t$ denotes the local truncation error.

Making use of Eq. (19) and Eq. (20), we obtain an error equation:

$$AE = \tau(h) \tag{21}$$

where $E = \bar{Y} - Y = (e_0, e_1, ..., e_N)^t$.

Let S_i be the sum of elements of the i^{th} row of A, then

$$S_1 = \sum_{j=1}^{N-1} m_{1j}, \text{ for } i = 1.$$

$$= \left(\frac{2\varepsilon}{h^2} + a_i + \frac{a_i^2 h^2}{12\varepsilon} + \frac{a_i^3 h^4}{360\varepsilon^2} + \frac{11 a_i^4 h^6}{226{,}800\varepsilon^3} \right) + \left(-\frac{\varepsilon}{h^2} + \frac{a_i^4 h^6}{1{,}814{,}400\varepsilon^3} \right)$$

Therefore, $S_1 = \frac{\varepsilon}{h^2} + a_i + \frac{a_i^2 h^2}{12\varepsilon} + \frac{a_i^3 h^4}{360\varepsilon^2} + \frac{89 a_i^4 h^6}{1{,}814{,}400\varepsilon^3}$, for $i = 1$

$$S_i = \sum_{j=1}^{N-1} m_{ij}, \quad \text{for } i = 2,3,..., N-2$$

$$= 2\left(-\frac{\varepsilon}{h^2} + \frac{a_i^4 h^6}{1{,}814{,}400\varepsilon^3} \right) + \left(\frac{2\varepsilon}{h^2} + a_i + \frac{a_i^2 h^2}{12\varepsilon} + \frac{a_i^3 h^4}{360\varepsilon^2} + \frac{11 a_i^4 h^6}{226{,}800\varepsilon^3} \right)$$

Therefore, $S_i = a_i + A_0 h^2$, where $A_0 = \frac{a_i^2}{12\varepsilon} + \frac{a_i^3 h^2}{360\varepsilon^2} + \frac{3 a_i^4 h^4}{181{,}440\varepsilon^3}$ for $|a_i| = \min_{2 \le i \le N-2} S_i$

$$S_{N-1} = \sum_{j=1}^{N-1} m_{N-1,j}, \text{ for } i = N-1.$$

$$= \left(-\frac{\varepsilon}{h^2} + \frac{a_i{}^4 h^6}{1{,}814{,}400\varepsilon^3} \right) + \left(\frac{2\varepsilon}{h^2} + a_i + \frac{a_i{}^2 h^2}{12\varepsilon} + \frac{a_i{}^3 h^4}{360\varepsilon^2} + \frac{11 a_i{}^4 h^6}{226{,}800\varepsilon^3} \right)$$

Therefore, $S_{N-1} = \frac{\varepsilon}{h^2} + a_i + \frac{a_i{}^2 h^2}{12\varepsilon} + \frac{a_i{}^3 h^4}{360\varepsilon^2} + \frac{89 a_i{}^4 h^6}{1{,}814{,}400\varepsilon^3}$ for $i = N-1$

Since $0 < \varepsilon << 1$, we can choose h sufficiently small so that the matrix A is irreducible and monotone [7]; Then it follows that A^{-1} exists and its elements are non-negative.
Hence, from Eq. (21), we get

$$E = A^{-1}.T(h) \tag{22}$$

and

$$\|E\| = \|A^{-1}\|.\|T(h)\| \tag{23}$$

Let $\overline{m}_{k,i}$ be the $(k,i)^{th}$ elements of A^{-1}. Since $\overline{m}_{k,i} \geq 0$, by the definition of multiplication of matrices with its inverses we have

$$\sum_{i=1}^{N-1} \overline{m}_{k,i}.S_i = 1, \ k = 1,2,...,N-1 \tag{24}$$

Therefore, it follows that

$$\sum_{i=1}^{N-1} \overline{m}_{k,i} \leq \frac{1}{\min\limits_{1 \leq i \leq N-1} S_i} = \frac{1}{|a_i|} \tag{25}$$

We define $\|A^{-1}\| = \max\limits_{1 \leq i \leq N-1} \sum\limits_{i=1}^{N-1} |\overline{m}_{k,i}|$ and $\|T(h)\| = \max\limits_{1 \leq i \leq N-1} |T_i(h_i)|$.

From Eqs. (20), (23) and (24) and (26), we obtain:

$$e_j = \sum_{i=1}^{N-1} \overline{m}_{k,i}.\, T_i(h), \ j = 1,...,N-1$$

$$e_j \leq \frac{1}{|a_i|}.T_i(h)$$

Therefore, $e_j \leq \frac{kh^{10}}{|a_i|}, \ j = 1,2,....,N-1$

Where, $k = \frac{a_i{}^4}{21{,}772{,}800\varepsilon^3} \left| y_i^{(4)} \right| + \frac{\varepsilon}{19{,}958{,}400} \left| y_i^{(12)} \right|$ which is a constant and independent of h.

Therefore, $\|E\| \leq o(h^{10})$.

This implies that the method gives a tenth order convergence.

4. Numerical Examples

To demonstrate the applicability of the methods, two model singularly perturbed problems have been considered. These examples have been chosen because they have been widely discussed in the literature and their exact solutions are available for comparison.

Example 1: Consider the following singular perturbation problem with constant coefficients:

$$-\varepsilon y'' + y = x,\ 0 \le x \le 1$$

with boundary conditions $y(0) = 1,\ y(1) = 1 + \exp\left(-\frac{1}{\sqrt{\varepsilon}}\right)$.

The exact (analytical) solution is given by:

$$y(x) = x + \exp\left(-\frac{x}{\sqrt{\varepsilon}}\right)$$

The numerical solutions in terms of maximum absolute errors and its comparison with other authors are tabulated in Table 1 for different values of ε and N.

Table 1 Maximum Absolute Error for Example 1.

ε	N = 16	N = 32	N = 64	N = 128	N = 256
Our method					
1/16	7.2164E-15	3.3307E-16	1.4433E-15	3.1086E-15	4.3299E-15
1/32	2.2993E-13	3.3307E-16	5.5511E-16	1.3323E-15	8.6597E-15
1/64	7.2617E-12	7.3275E-15	5.5511E-16	9.9920E-16	3.3307E-15
1/128	2.1337E-10	2.3015E-13	2.7756E-16	8.8818E-16	1.6653E-15
1/256	6.7755E-09	7.2617E-12	7.2720E-15	3.8858E-16	4.5075E-14
Fasika et al. [4]					
1/16	8.0337E-09	1.2628E-10	1.9704E-12	2.6756E-14	3.1575E-13
1/32	6.4174E-08	1.0146E-09	1.5920E-11	2.7744E-13	3.3595E-13
1/64	5.0661E-07	8.1031E-09	1.2737E-10	1.9886E-12	4.4076E-14
1/128	3.7264E-06	6.4204E-08	1.0151E-09	1.5928E-11	2.7062E-13
1/256	2.9689E-05	5.0662E-07	8.1032E-09	1.2737e-10	1.9936e-12
Arshad and Pooja [10]					
1/16	2.153E-08	1.082E-10	1.536E-12	2.942E-14	2.522E-13
1/32	2.629E-07	1.591E-09	1.130E-11	1.844E-13	1.331E-13
1/64	2.832E-06	2.150E-08	1.081E-10	1.601E-12	4.596E-14
1/128	2.591E-05	2.629E-07	1.591E-09	1.130E-11	1.907E-13
1/256	1.922E-04	2.832E-06	2.150E-08	1.081E-10	1.599E-12

Example 2: Consider the following singular perturbation problem with constant coefficients:

$$-\varepsilon y'' + y = -\cos^2(\pi x) - 2\varepsilon\pi^2 \cos(2\pi x),\quad 0 \le x \le 1$$

with boundary conditions $y(0) = 0 = y(1)$

The exact (analytical) solution is given by:

$$y(x) = \frac{\exp(-(1-x)/\sqrt{\varepsilon}) + \exp(-x/\sqrt{\varepsilon})}{1 + \exp(-1/\sqrt{\varepsilon})} - \cos^2(\pi x)$$

The numerical solutions in terms of maximum absolute errors and its comparison with other authors are tabulated in Table 2 for different values of ε and N.

Table 2: Maximum Absolute Errors for Example 2

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$
Our method					
1/16	1.6536E-12	1.7208E-15	4.7184E-16	5.4123E-16	1.7208E-15
1/32	1.3279E-12	1.2906E-15	4.4409E-16	6.1062E-16	1.7764E-15
1/64	7.1339E-12	7.1609E-15	2.2204E-16	8.3267E-16	2.1094E-15
1/128	1.1333E-10	2.3015E-13	3.3307E-16	3.3307E-16	1.3323E-15
1/256	6.7755E-09	7.2617E-12	7.1054E-15	5.5511E-16	3.1641E-14
Fasika et al [4]					
1/16	3.1216E-07	4.8731E-09	7.6124E-11	1.1864E-12	6.5059E-14
1/32	2.6300E-07	4.1289E-09	6.4591E-11	1.0078E-12	8.6264E-14
1/64	4.7141E-07	7.5487E-09	1.1898E-10	1.8661E-12	4.8017E-14
1/128	3.6957E-06	6.3668E-08	1.0067E-09	1.5801E-11	2.3137E-13
1/256	2.9667E-05	5.0624E-07	8.0973E-09	1.2727E-10	1.9914E-12
Arshad and Pooja [10]					
1/16	4.707E-07	5.254E-09	7.265E-11	1.089E-12	5.001E-14
1/32	2.681E-07	3.897E-09	5.920E-11	9.230E-13	2.559E-14
1/64	2.603E-06	1.908E-08	9.803E-11	1.502E-12	4.968E-14
1/128	2.560E-05	2.607E-07	1.581E-09	1.119E-11	2.405E-13
1/256	1.920E-04	2.830E-06	2.149E-08	1.080E-10	1.579E-12

Example 3: Consider the following singular perturbation problem with constant coefficients:

$$-\varepsilon y''(x)+y(x)=1+2\sqrt{\varepsilon}\{\exp\left(\frac{x-1}{\sqrt{\varepsilon}}\right)+\exp\left(\frac{-x}{\sqrt{\varepsilon}}\right)\}$$

with boundary conditions $y(0)=0=y(1)$

The exact (analytical) solution of the above problem is:

$$y(x)=1-(1-x)\exp\left(\frac{-x}{\sqrt{\varepsilon}}\right)-x\exp\left(\frac{x-1}{\sqrt{\varepsilon}}\right)$$

The numerical solutions in terms of maximum absolute errors and its comparison with other method are tabulated in Table 3 for different values of ε and N.

Table 3: Maximum Absolute Errors for Example 3

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$
Our method					
1/16	3.7415E-14	5.5511E-16	1.8874E-15	3.8858E-15	7.4385E-15
1/32	7.1430E-13	8.8818E-16	4.4409E-16	1.7764E-15	4.4409E-15
1/64	1.7440E-11	1.7319E-14	8.8818E-16	1.9984E-15	4.4409E-15
1/128	4.2294E-10	4.5353E-13	4.4409E-16	8.8818E-16	2.3315E-15
1/256	1.1398E-08	1.2244E-11	1.2212E-14	9.9920E-16	5.6954E-14
Fasika et al [4]					
1/16	3.0275 E -08	4.7605 E -10	7.4485 E -12	1.1768 E -13	3.5527 E -15
1/32	1.5650 E -07	2.4759 E -09	3.8809 E -11	6.0718 E -13	1.1990 E -14
1/64	9.5977 E -07	1.5367 E -08	2.4161 E -10	3.7805 E -12	5.9619 E -14
1/128	6.0692 E -06	1.0376 E -07	1.6413 E -09	2.5820 E -11	4.0390 E -13
1/256	4.2534 E -05	7.2759 E -07	1.1646 E -08	1.8308 E -10	2.8649 E -12
Arshad and Pooja [10]					
1/16	6.409E-08	3.191E-10	6.235E-12	1.529E-13	2.813E-13
1/32	6.085E-07	3.754E-09	2.810E-11	5.647E-13	1.972E-13
1/64	5.331E-06	4.174E-08	2.139E-10	2.972E-12	1.043E-13
1/128	4.120E-05	4.329E-07	2.673E-09	1.794E-11	3.867E-13
1/256	2.669E-04	4.077E-06	3.162E-08	1.610E-10	2.255E-12

5. Discussion and Conclusion

The tenth order compact finite difference method has been presented for solving singularly perturbed reaction-diffusion equations with dirichlet boundary conditions. Derivatives appearing in the given differential equation are replaced by finite difference approximations obtained by Taylor series expansions at the grid points. This gives a large algebraic tri-diagonal system of equations to be solved by Thomas algorithm, and to obtain the solutions at the mesh points using MATLAB software. Three model examples are given to demonstrate the efficiency of the proposed method. The maximum absolute errors tabulated in (Tables 1 – 3) for different values of the perturbation parameter ε and mesh size h are compared with some previous findings of other methods reported in the literature. As it can be observed from the tables, the proposed method improved the findings reported by authors' given in [4] and [10].

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