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Several Stress Resultant and Deflection Formulas for Euler-Bernoulli Beams under Concentrated and Generalized Power/Sinusoidal Distributed Loads

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Abstract

In the present paper, the transfer matrix method based on the Euler-Bernoulli beam theory is exploited to originally achieve some exact analytical formulas for classically supported beams under both the concentrated and generalized power/sinusoidal distributed loads. A general solution procedure is also presented to consider different loads and boundary conditions. Those closed-form formulas can be used in a variety of engineering applications as well as benchmark solutions.

Keywords: Transfer matrix method, initial value problem, exact solution, Euler-Bernoulli beam, distributed loads.

1. Introduction

As is well known Euler-Bernoulli beam theory called classical beam theory is founded on the following assumptions: i) The cross section of the beam does not significantly deform under applied loads and can be assumed as rigid, ii) The cross section of the beam remains planar and normal to the deformed axis of the beam during the deformation. Due to the assumptions given above, in Euler-Bernoulli beams, which are very good for thin beam applications, transverse shear stress is not taken into account contrary to Timoshenko beams, which are good for thick beams. In Timoshenko beams the cross-section remains planar but does not remain normal to the neutral axis after bending. The basis of Euler-Bernoulli beam theory are well introduced in the text books in engineering educational system. There are also some engineering handbooks which cover Euler-Bernoulli exact solutions of many certain types of problems [1-3]. The present study aims at adding some remarkable closed-form formulas to the deep open repository for Euler-Bernoulli beam bending formulas. To this end the transfer matrix approach which is one of the initial value problem (IVP) solver methods is employed [4-6].

2. Application of the Transfer Matrix Method

Let x be the beam axis (Fig. 1). The governing homogeneous differential equation set for the out-of-plane bending analysis of the beam having uniform section in canonical form is given by [4]

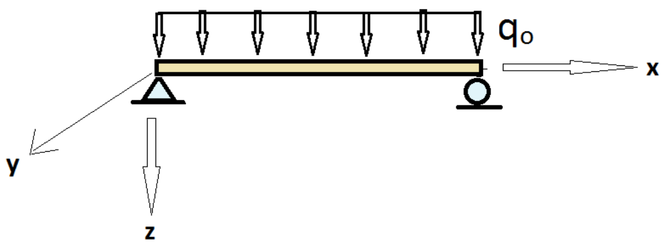


Fig. 1. A beam under uniformly distributed forces

$$\frac{dw(x)}{dx} = -\theta(x), \qquad \frac{d\theta(x)}{dx} = \frac{M(x)}{EI}, \qquad \frac{dM(x)}{dx} = T(x), \qquad \frac{dT(x)}{dx} = 0 \tag{1}$$

where $w(x)$ is the transverse displacement, $\theta(x)$ is the rotation, $M(x)$ is the bending moment, and $T(x)$ is the shear force. By using the prime symbol for the derivative of the related quantity with respect to x, Eq. (1) may be written in a compact form as

$$\boldsymbol{S}'(x) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{EI} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} w(x) \\ \theta(x) \\ M(x) \\ T(x) \end{Bmatrix} = \boldsymbol{D}\,\boldsymbol{S}(x) \tag{2}$$

where $\boldsymbol{S}(x)$ is called the state vector which comprises the cross-sectional quantities at a positive section, and $\boldsymbol{D}$ is the differential matrix. Characteristic equation of the differential matrix is

$$|\boldsymbol{D} - \lambda \boldsymbol{I}| = \lambda^4 = 0 \tag{3}$$

where $\boldsymbol{I}$ refers the unit matrix. Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation, and so $\boldsymbol{D}^4 = 0$. Equation (3) suggests that the higher powers of the differential matrix which are equal or greater than four are identically zero. The transfer matrix satisfies the similar type of differential equation for the state vector given in Eq. (2),
$\boldsymbol{F}'(x) = \boldsymbol{D}\,\boldsymbol{F}(x)$. If the elements of the differential matrix are constants as in Eq. (2), it is possible to get an exact solution to the transfer matrix. In this case, solution of $\boldsymbol{F}'(x) = \boldsymbol{D}\,\boldsymbol{F}(x)$ with the initial conditions, $\boldsymbol{F}(x = 0) = \boldsymbol{I}$, gives us the exact transfer matrix in the form of a matrix exponential

$$\boldsymbol{F}(x) = e^{x\boldsymbol{D}} = \boldsymbol{I} + x\boldsymbol{D} + \frac{x^2}{2!}\boldsymbol{D}^2 + \frac{x^3}{3!}\boldsymbol{D}^3 \tag{4}$$

In the series solution in Eq. (4), the remaining terms including forth and higher than forth powers of the differential matrix are taken as identical to the zero since $\boldsymbol{D}^4 = 0$.

$$\boldsymbol{F}(x) = \begin{bmatrix} 1 & -x & -\frac{x^2}{2EI} & -\frac{x^3}{6\mathrm{EI}} \\ 0 & 1 & \frac{x}{\mathrm{EI}} & \frac{x^2}{2\mathrm{EI}} \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

Suppose that a beam is subjected to both a distributed force $q(x)$ and a distributed couple moment $m(x)$ along the beam axis. together with a concentrated force P_o and a couple moment μ_o acting at section $x = a$. Under this assumption, the overall transfer matrix relates the state vectors at both ends of the beam as follows

$$\boldsymbol{S}(L) = \boldsymbol{F}(L)\boldsymbol{S}(0) + \int_0^L \boldsymbol{F}(L-\xi)\boldsymbol{k}(\xi)\,\mathrm{d}\xi + \boldsymbol{F}(L-a)\boldsymbol{K}(a) \tag{6}$$

where

$$\boldsymbol{S}(L) = \begin{Bmatrix} w(L) \\ \theta(L) \\ M(L) \\ T(L) \end{Bmatrix} = \begin{Bmatrix} w_L \\ \theta_L \\ M_L \\ T_L \end{Bmatrix} \tag{7}$$

$$\boldsymbol{F}(L-\xi) = \begin{bmatrix} 1 & \xi - L & -\frac{(L-\xi)^2}{2EI} & -\frac{(L-\xi)^3}{6EI} \\ 0 & 1 & \frac{L-\xi}{EI} & \frac{(L-\xi)^2}{2EI} \\ 0 & 0 & 1 & L-\xi \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

$$\boldsymbol{S}(0) = \begin{Bmatrix} w(0) \\ \theta(0) \\ M(0) \\ T(0) \end{Bmatrix} = \begin{Bmatrix} w_o \\ \theta_o \\ M_o \\ T_o \end{Bmatrix} \tag{9}$$

$$\boldsymbol{k}(\xi) = \begin{Bmatrix} 0 \\ 0 \\ -m(\xi) \\ -q(\xi) \end{Bmatrix} \tag{10}$$

and

$$\boldsymbol{K}(a) = \begin{Bmatrix} 0 \\ 0 \\ -\mu_o \\ -P_o \end{Bmatrix} \tag{11}$$

In Eq. (6), column matrix $\boldsymbol{k}(\xi)$ signifies the nonhomogeneous solution due to the distributed forces, and $\boldsymbol{K}(a)$ is referred to as a discontinuity matrix due to the concentrated intermediate loads. By letting

$$\boldsymbol{\chi} = \int_0^L \boldsymbol{F}(L-\xi)\boldsymbol{k}(\xi)\,\mathrm{d}\xi, \qquad \boldsymbol{\kappa} = \boldsymbol{F}(L-a)\boldsymbol{K}(a) \tag{12}$$

Eq. (6) reads

$$\begin{aligned}
w_L &= \mathrm{F(L)}_{1,1}w_o + \mathrm{F(L)}_{1,2}\theta_o + \mathrm{F(L)}_{1,3}M_o + \mathrm{F(L)}_{1,4}T_o + \chi_1 + \kappa_1 \\
\theta_L &= \mathrm{F(L)}_{2,1}w_o + \mathrm{F(L)}_{2,2}\theta_o + \mathrm{F(L)}_{2,3}M_o + \mathrm{F(L)}_{2,4}T_o + \chi_2 + \kappa_2 \\
M_L &= \mathrm{F(L)}_{3,1}w_o + \mathrm{F(L)}_{3,2}\theta_o + \mathrm{F(L)}_{3,3}M_o + \mathrm{F(L)}_{3,4}T_o + \chi_3 + \kappa_3 \\
T_L &= \mathrm{F(L)}_{4,1}w_o + \mathrm{F(L)}_{4,2}\theta_o + \mathrm{F(L)}_{4,3}M_o + \mathrm{F(L)}_{4,4}T_o + \chi_4 + \kappa_4
\end{aligned} \tag{13}$$

Boundary conditions for the beam considered in the present study is given in Table 1. In the transfer matrix method, it is necessary to determine all the elements of the initial state vector to get a general solution to the problem. Some of elements of the initial state vector may be given directly as boundary conditions. To find the remaining unknown ones, the boundary conditions given at both ends should be implemented into Eq. (18) by considering Table 1. After determining of the full elements of the initial state vector, all sectional quantities at any section may be easily computed as follows

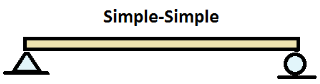
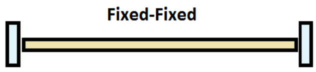
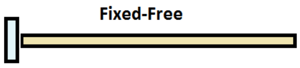
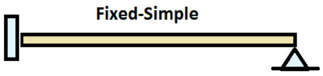
$$For\ (0 \le x < a), \quad \boldsymbol{S^I}(\mathrm{x}) = \boldsymbol{F}(x)\boldsymbol{S}(0) + \int_0^x \boldsymbol{F}(x-\xi)\boldsymbol{k}(\xi)\,\mathrm{d}\xi \tag{14}$$

$$For\ (a \le x \le L), \quad \boldsymbol{S^{II}}(\mathrm{x}) = \boldsymbol{F}(x)\boldsymbol{S}(0) + \int_0^x \boldsymbol{F}(x-\xi)\boldsymbol{k}(\xi)\,\mathrm{d}\xi + \boldsymbol{F}(x-a)\boldsymbol{K}(a)$$

If there are more than one discontinuities along the beam axis the following may be observed [4].

$$\mathrm{S(x)} = \boldsymbol{F}(x)\boldsymbol{S}(0) + \int_0^x \boldsymbol{F}(x-\xi)\boldsymbol{k}(\xi)\,\mathrm{d}\xi + \sum_{i=1}^{n} \boldsymbol{F}(x-\xi_i)\boldsymbol{K}(\xi_i) \tag{15}$$

Table 1. Boundary conditions considered

	Classically supported beams	$x=0$	$x=L$
Simple-Simple (S-S)		$w_o=0,\ M_o=0$	$w_L=0, M_L=0$
Clamped-Clamped (C-C)		$w_o=0,\ \theta_o=0$	$w_L=0,\ \theta_L=0$
Clamped-Free (C-F)		$w_o=0, \theta_o=0$	$T_L=0,\ M_L=0$
Clamped-Simple (C-S)		$w_o=0, \theta_o=0$	$w_L=0, M_L=0$

In the following sections the analytical formulas are to be presented for beams under separate distributed and concentrated loads. Since small deformations are assumed, the superposition principle is hold when necessary.

3. Solutions for Uniformly Distributed Forces

For only uniformly distributed forces and couple moments acting along the beam,

$$q(x) = -q_o, \quad m(x) = -m_d \tag{16}$$

a general solution takes the following form $(0 \leq x \leq L)$

$$\mathbf{S}(\mathrm{x}) = \boldsymbol{F}(x)\boldsymbol{S}(0) + \int_0^x \boldsymbol{F}(x-\xi)\boldsymbol{k}(\xi)\,d\xi = \boldsymbol{F}(x)\boldsymbol{S}(0) + \begin{Bmatrix} \dfrac{x^3(4m_d + xq_o)}{24\mathrm{EI}} \\ -\dfrac{x^2(3m_d + xq_o)}{6\mathrm{EI}} \\ -\dfrac{1}{2}x(2m_d + xq_o) \\ -xq_o \end{Bmatrix} \tag{17}$$

3.1. S-S Beam under Uniformly Distributed Loads

Distribution of stress resultants, displacements and rotations in a simply supported beam are found as

$$\boldsymbol{S}_{S-S}^{E}(\mathrm{x}) = \begin{Bmatrix} w(x) \\ \theta(x) \\ M(x) \\ T(x) \end{Bmatrix}_{S-S} = \boldsymbol{F}(x)\boldsymbol{S}(0) + \int_0^{\boldsymbol{x}} \boldsymbol{F}(x-\xi)\boldsymbol{k}(x)\,d\xi$$

$$= \boldsymbol{F}(x) \begin{Bmatrix} 0 \\ -\dfrac{L^3 q_o}{24\mathrm{EI}} \\ 0 \\ m_d + \dfrac{Lq_o}{2} \end{Bmatrix} + \begin{Bmatrix} \dfrac{x^3(4m_d + xq_o)}{24\mathrm{EI}} \\ -\dfrac{x^2(3m_d + xq_o)}{6\mathrm{EI}} \\ -\dfrac{1}{2}x(2m_d + xq_o) \\ -xq_o \end{Bmatrix} = \begin{Bmatrix} \dfrac{x(L^3 - 2Lx^2 + x^3)q_o}{24\mathrm{EI}} \\ -\dfrac{(L^3 - 6Lx^2 + 4x^3)q_o}{24\mathrm{EI}} \\ \dfrac{1}{2}x(L-x)q_o \\ m_d + \dfrac{1}{2}(L-2x)q_o \end{Bmatrix} \tag{18}$$

For the sake of comparison the followings values at specific sections may be used.

$$w_{L/2} = \frac{5L^4 q_o}{384\mathrm{EI}}$$

$$\theta_o = -\frac{L^3 q_o}{24\mathrm{EI}}, \qquad \theta_{L/2} = 0, \qquad \theta_L = \frac{L^3 q_o}{24\mathrm{EI}} \tag{19a}$$

$$M_{L/2} = \frac{L^2 q_o}{8}$$

$$T_o = m_d + \frac{Lq_o}{2}, \quad T_{L/2} = m_d, \quad T_L = m_d - \frac{Lq_o}{2} \tag{19b}$$

Comparison of Bernoulli-Euler and Timoshenko beam's dimensionless displacements, $\overline{w}$, when $m_d = 0$ is shown in Fig. 2.

$$\overline{w} = \frac{EI}{q_o L^4} w \tag{20}$$

For $L/h = 10, 20, 50,$ and 100, the maximum displacements in Euler beam remain constant as $\overline{w}_{max}^{E} = 0.0130208$ while it takes different values in Timoshenko beams as $\overline{w}_{max}^{T} = 0.0133458,\ 0.0131021,\ 0.0130338,$ and 0.0130241, respectively. The maximum displacements in a S-S Euler beam is found as $\overline{w}_{max}^{E}$ =0.013130 in Ref. [7], as $\overline{w}_{max}^{E}$ =0.013152 in Ref. [8], and as $\overline{w}_{max}^{E} = 0.0130208$ in Ref. [9].

It is worth noting that there is no difference in the values of rotation, bending moment, and shearing force in S-S beams subjected to a uniform distributed force along the beam based on the two beam theories. From Fig. 3, it is observed that Timoshenko's beam theory gives somewhat higher displacements in S-S beams than Euler-Bernoulli beam theory.

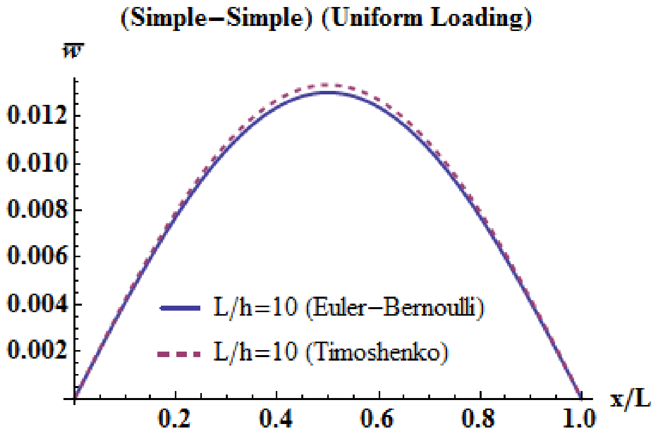


Fig. 2. Dimensionless transverse displacements in a S-S beam based on the two beam theories

3.2. C-S Beam under Uniformly Distributed Loads

Variations of stress resultants, displacements and rotations in a fixed-simple supported Euler-Bernoulli beam along the beam axis are

$$\boldsymbol{S}_{C-S}^{E}(\mathrm{x}) = \begin{Bmatrix} w(x) \\ \theta(x) \\ M(x) \\ T(x) \end{Bmatrix}_{C-S} = \boldsymbol{F}(x)\boldsymbol{S}(0) + \int_0^x \boldsymbol{F}(x-\xi)\boldsymbol{k}(x)\, d\xi \tag{21a}$$

$$= \boldsymbol{F}(x) \begin{Bmatrix} 0 \\ 0 \\ -\frac{1}{8}L^2 q_o \\ m_d + \frac{5Lq_o}{8} \end{Bmatrix} + \begin{Bmatrix} \frac{x^3(4m_d + xq_o)}{24\mathrm{EI}} \\ -\frac{x^2(3m_d + xq_o)}{6\mathrm{EI}} \\ -\frac{1}{2}x(2m_d + xq_o) \\ -xq_o \end{Bmatrix} = \begin{Bmatrix} \frac{x^2(3L-2x)(L-x)q_o}{48\mathrm{EI}} \\ -\frac{x(6L^2 - 15Lx + 8x^2)q_o}{48\mathrm{EI}} \\ -\frac{1}{8}(L-4x)(L-x)q_o \\ m_d + \frac{1}{8}(5L-8x)q_o \end{Bmatrix}$$

(21b)

Some specific values of Eq. (21) are as follows

$$w_{L/2} = \frac{L^4 q_o}{192\text{EI}}$$
$$\theta_o = 0, \qquad \theta_{L/2} = -\frac{L^3 q_o}{192\text{EI}}, \qquad \theta_L = \frac{L^3 q_o}{48\text{EI}}$$
$$M_o = -\frac{1}{8}L^2 q_o, \quad M_{L/2} = \frac{L^2 q_o}{16}, \qquad M_L = 0 \qquad (22)$$
$$T_o = m_d + \frac{5Lq_o}{8}, \quad T_{L/2} = m_d + \frac{Lq_o}{8}, \quad T_L = m_d - \frac{3Lq_o}{8}$$

Elementary theory states that the maximum displacement occurs approximately at $x = (1 - 0.421535)L$ [3]. For $L/h = 10, 20, 50,$ *and* 100, the dimensionless displacements in Euler beam at the section of $x/L = 0.6$ remain constant as $\overline{w}^E_{x/L=0.6} = 0.0054$ while it takes different values in Timoshenko beams as $\overline{w}^T_{x/L=0.6} = 0.00576618$, 0.00549162, 0.00541466, and 0.00540367, respectively. The transverse deflection was found as $\overline{w}^E_{x/L=0.5} = 0.00520833$ in both the present study and in Ref. [9]. It may be noted that there are also differences in the bending moment, shearing force, and the rotation in a C-S beam based on the two beam theories. Figure 3 shows the dimensionless transverse displacements in a C-S beam based on the two beam theories. In a C-S beam the differences in the results of the two beam theories become much clearer than S-S beam.

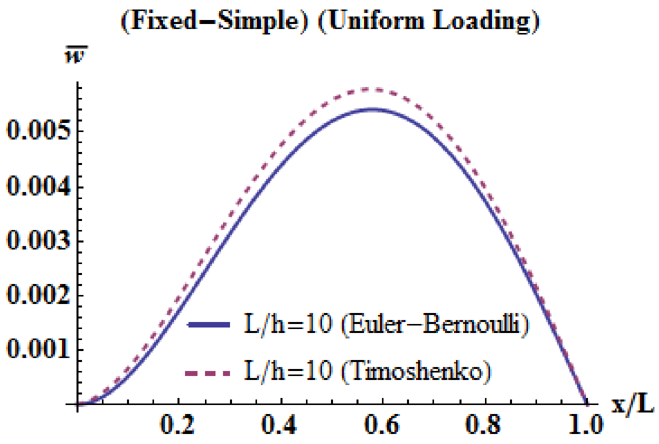


Fig. 3. Dimensionless transverse displacements in a C-S beam based on the two beam theories

3.3. C-F Beam under Uniformly Distributed Loads

Stress resultants, displacements and rotations in a fixed-free supported beam vary along the beam axis as

$$\boldsymbol{S}^E_{C-F}(\text{x}) = \begin{Bmatrix} w(x) \\ \theta(x) \\ M(x) \\ T(x) \end{Bmatrix}_{C-F} = \boldsymbol{F}(x)\boldsymbol{S}(0) + \int_0^x \boldsymbol{F}(x-\xi)\boldsymbol{k}(x)\, d\xi \qquad (23a)$$

$$\boldsymbol{S}_{C-F}^{E}(\mathrm{x}) = \boldsymbol{F}(x)\begin{Bmatrix} 0 \\ 0 \\ Lm_d - \dfrac{L^2 q_o}{2} \\ Lq_o \end{Bmatrix} + \begin{Bmatrix} \dfrac{x^3(4m_d + xq_o)}{24\mathrm{EI}} \\ -\dfrac{x^2(3m_d + xq_o)}{6\mathrm{EI}} \\ -\dfrac{1}{2}x(2m_d + xq_o) \\ -xq_o \end{Bmatrix}$$

$$= \begin{Bmatrix} \dfrac{x^2(4m_d(x-3L) + (6L^2 - 4Lx + x^2)q_o)}{24\mathrm{EI}} \\ -\dfrac{x(3m_d(x-2L) + (3L^2 - 3Lx + x^2)q_o)}{6\mathrm{EI}} \\ \dfrac{1}{2}(L-x)(2m_d + (x-L)q_o) \\ (L-x)q_o \end{Bmatrix} \tag{23b}$$

Certain values of sectional quantities in a C-F beam are

$$w_{L/2} = \frac{L^3(17Lq_o - 40m_d)}{384\mathrm{EI}}, \quad w_L = \frac{L^3(3Lq_o - 8m_d)}{24\mathrm{EI}}$$

$$\theta_{L/2} = \frac{L^2(18m_d - 7Lq_o)}{48\mathrm{EI}}, \quad \theta_L = -\frac{L^2(Lq_o - 3m_d)}{6\mathrm{EI}} \tag{24}$$

$$M_o = Lm_d - \frac{L^2 q_o}{2}, \quad M_{L/2} = -\frac{1}{8}L(Lq_o - 4m_d)$$

$$T_o = Lq_o, \quad T_{L/2} = \frac{Lq_o}{2}, T_L = 0$$

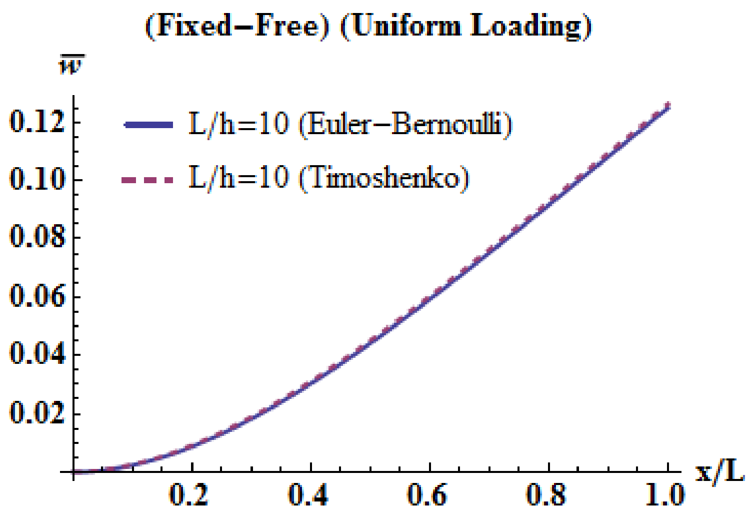


Fig. 4. Dimensionless transverse displacements in a C-F beam based on the two beam theories

Dimensionless transverse displacements in a C-F beam based on the two beam theories is illustrated in Fig. 4 when the beam is subjected only distributed uniform forces. Euler-Bernoulli displacements are again independent from L/h ratios. The maximum displacement in an Euler-Bernoulli beam is calculated at the free end as $\overline{w}_L^E = 0.125$. In Timoshenko beams, those values are to be $\overline{w}_L^T = 0.1263$ $(L/h = 10)$, 0.125325 $(L/h = 20)$, 0.125052 $(L/h = 50)$, and 0.125013 $(L/h = 100)$. Similar to the S-S beam, there is no difference between the results of two beam theories for the rotations, bending moments and shear forces in a C-F beam.

3.4. C-C Beam under Uniformly Distributed Loads

Let's consider a fixed-fixed beam. Analytical formulas derived are as follows

$$\boldsymbol{S}_{C-C}^{E}(\mathrm{x}) = \begin{Bmatrix} w(x) \\ \theta(x) \\ M(x) \\ T(x) \end{Bmatrix}_{C-C} = \boldsymbol{F}(x)\boldsymbol{S}(0) \int_0^x \boldsymbol{F}(x-\xi)\boldsymbol{k}(x)\, d\xi$$

$$= \boldsymbol{F}(x) \begin{Bmatrix} 0 \\ 0 \\ -\dfrac{1}{12}L^2 q_o \\ \dfrac{Lq_o}{2} \end{Bmatrix} + \begin{Bmatrix} \dfrac{x^3(4m_d + xq_o)}{24\mathrm{EI}} \\ -\dfrac{x^2(3m_d + xq_o)}{6\mathrm{EI}} \\ -\dfrac{1}{2}x(2m_d + xq_o) \\ -xq_o \end{Bmatrix} = \begin{Bmatrix} \dfrac{x^2(L-x)^2 q_o}{24\mathrm{EI}} \\ -\dfrac{x(L-2x)(L-x)q_o}{12\mathrm{EI}} \\ -\dfrac{1}{12}(L^2 - 6Lx + 6x^2)q_o \\ m_d + \dfrac{1}{2}(L-2x)q_o \end{Bmatrix} \tag{25}$$

Selected values of sectional quantities in a C-C beam are

$$\begin{gathered} w_{L/2} = \frac{L^4 q_o}{384\mathrm{EI}} \\ \theta_o = \theta_{L/2} = \theta_L = 0 \\ M_o = -\frac{1}{12}L^2 q_o,\ M_{L/2} = \frac{L^2 q_o}{24},\ M_L = -\frac{1}{12}L^2 q_o \\ T_o = m_d + \frac{Lq_o}{2},\ T_{L/2} = m_d,\ T_L = m_d - \frac{Lq_o}{2} \end{gathered} \tag{26}$$

Variation of the dimensionless transverse displacement in a C-C beam is demonstrated in Fig. 5 for $q(x) = q_o$. As seen from Fig. 5, the maximum dimensionless transverse displacement in a C-C beam occurs at the mid-span of the beam. Based on the Euler-Bernoulli beam theory, the maximum displacement is evaluated as $\overline{w}_{max}^E = 0.00260417$. In Timoshenko beams, these values are to be $\overline{w}_{max}^T = 0.00292917$ $(L/h = 10)$, 0.00268542 $(L/h = 20)$, 0.00261717 $(L/h = 50)$, and $0.00260742 (L/h = 100)$. Similar to the S-S and C-F beams, there is no difference between the results of two beam theories for the rotations, bending moments and shear forces in a C-C beam under uniformly distributed forces.

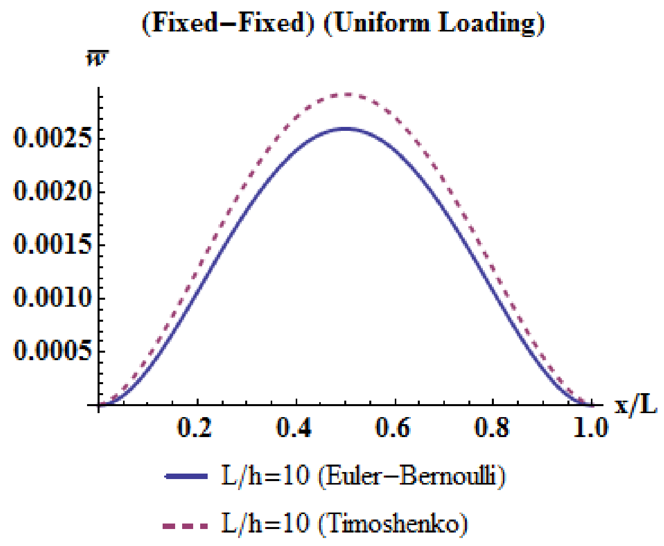


Fig. 5. Dimensionless transverse displacements in a C-C beam based on the two beam theories

4. Solutions for Sinusoidal Distributed Forces

A generalized sinusoidal distributed force [10] may be in the form of (Fig. 6)

$$q(x) = -q_o \sin\left(\frac{n\pi x}{L}\right), \qquad n \geq 0 \tag{27}$$

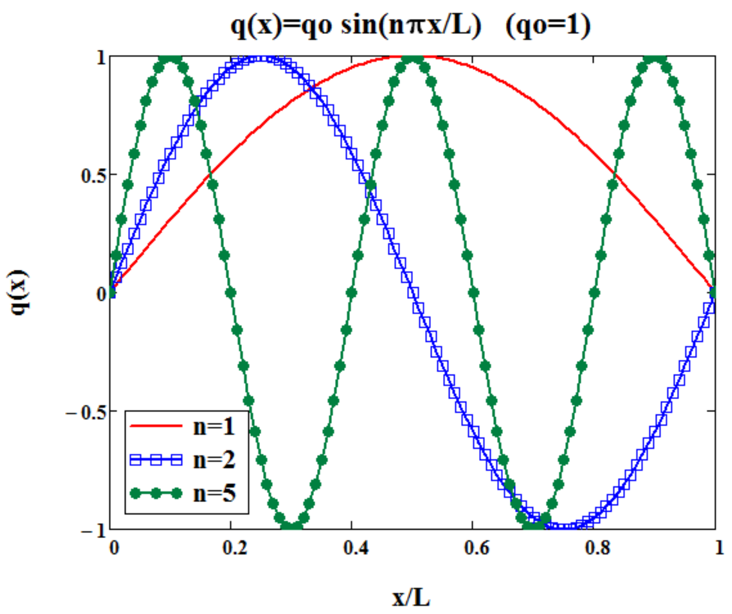


Fig. 6. Generalized sinusoidal loading

In the case of sinusoidal distributed forces in Eq. (27), the particular solution becomes

$$\int_0^x \boldsymbol{F}(x-\xi)\boldsymbol{k}(\xi)\,d\xi = \left\{\begin{array}{c} \frac{Lq_o\left(6L^3\sin\left(\frac{\pi nx}{L}\right)-6\pi L^2nx+\pi^3n^3x^3\right)}{6\pi^4\mathrm{EI}n^4} \\ -\frac{Lq_o\left(2L^2\cos\left(\frac{\pi nx}{L}\right)-2L^2+\pi^2n^2x^2\right)}{2\pi^3\mathrm{EI}n^3} \\ \frac{Lq_o\left(\mathrm{L}\sin\left(\frac{\pi nx}{L}\right)-\pi nx\right)}{\pi^2n^2} \\ \frac{Lq_o\left(\cos\left(\frac{\pi nx}{L}\right)-1\right)}{\pi n} \end{array}\right\} \quad (28)$$

A general solution takes the form of $\mathbf{S}(\mathrm{x}) = \boldsymbol{F}(x)\boldsymbol{S}(0) + \int_0^x \boldsymbol{F}(x-\xi)\boldsymbol{k}(\xi)\,d\xi$.

4.1. S-S Beam under Sinusoidal Distributed Loads

The rotation of the section about y- axis and the shearing force at the initial end is found as $(w_o = 0, M_o = 0)$

$$\theta_o = \frac{L^3((\pi^2n^2+6)\sin(\pi n)-6\pi n)q_o}{6\pi^4\mathrm{EI}n^4}$$
$$T_o = \frac{L(\pi n-\sin(\pi n))q_o}{\pi^2n^2} \quad (29)$$

With the help of Eq. (29), the general solution in a closed form is obtained for simply supported beam under a general sinusoidal load as follows

$$\boldsymbol{S}(x)_{S-S}^{E} = \left\{\begin{array}{c} \frac{Lq_o\left(\begin{array}{c}6L^3\sin\left(\frac{\pi nx}{L}\right)\\ +\mathrm{x}\sin(\pi n)\left(\pi^2n^2x^2-L^2(\pi^2n^2+6)\right)\end{array}\right)}{6\pi^4\mathrm{EI}n^4} \\ \frac{Lq_o\left(\begin{array}{c}\sin(\pi n)\left(L^2(\pi^2n^2+6)-3\pi^2n^2x^2\right)\\ -6\pi L^2\,n\cos\left(\frac{\pi nx}{L}\right)\end{array}\right)}{6\pi^4\mathrm{EI}n^4} \\ \frac{Lq_o\left(\mathrm{L}\sin\left(\frac{\pi nx}{L}\right)-\mathrm{x}\sin(\pi n)\right)}{\pi^2n^2} \\ \frac{Lq_o\left(\pi\,n\cos\left(\frac{\pi nx}{L}\right)-\sin(\pi n)\right)}{\pi^2n^2} \end{array}\right\} \quad (30)$$

As may be guessed, in a S-S beam, the variation of $\theta(x), M(x),$ and $T(x)$ remain unchanged in both beam theories. Some chosen values of the sectional quantities are

$$w_{L/2} = \frac{L^4\left(\begin{array}{c}16\sin\left(\frac{\pi n}{2}\right)\\ -(\pi^2n^2+8)\sin(\pi n)\end{array}\right)q_o}{16\pi^4\mathrm{EI}n^4}$$

$$\theta_{L/2}=\frac{L^3 q_o\begin{pmatrix}(\pi^2n^2+24)\sin(\pi n)\\ -24\pi\,\mathrm{n}\cos\left(\frac{\pi n}{2}\right)\end{pmatrix}}{24\pi^4\mathrm{EI}n^4},\quad \theta_L=-\frac{L^3 q_o\begin{pmatrix}(\pi^2n^2-3)\sin(\pi n)\\ +3\pi\,\mathrm{n}\cos(\pi n)\end{pmatrix}}{3\pi^4\mathrm{EI}n^4} \tag{31}$$

$$M_{L/2}=\frac{4L^2\sin^3\left(\frac{\pi n}{4}\right)\cos\left(\frac{\pi n}{4}\right)q_o}{\pi^2n^2}$$

$$T_{L/2}=\frac{Lq_o\left(\pi\,\mathrm{n}\cos\left(\frac{\pi n}{2}\right)-\sin(\pi n)\right)}{\pi^2n^2},\quad T_L=\frac{Lq_o(\pi\,\mathrm{n}\cos(\pi n)-\sin(\pi n))}{\pi^2n^2}$$

If n is a positive integer ($\sin(\pi n)=0$), then Eq. (30) turns to be

$$\boldsymbol{S}(x)^E_{S-S(n=positive\ integer)}=\begin{Bmatrix}\dfrac{q_oL^4\sin\left(\frac{\pi nx}{L}\right)}{\pi^4\mathrm{EI}n^4}\\ -\left(\dfrac{q_o\pi L^3\,\mathrm{n}\cos\left(\frac{\pi nx}{L}\right)}{\pi^4\mathrm{EI}n^4}\right)\\ \dfrac{L^2q_o\sin\left(\frac{\pi nx}{L}\right)}{\pi^2n^2}\\ \dfrac{Lq_o\pi\,\mathrm{n}\cos\left(\frac{\pi nx}{L}\right)}{\pi^2n^2}\end{Bmatrix} \tag{32}$$

where $w^E_{S-S(n=positive\ integer)}=q_oL^4\sin\left(\frac{\pi nx}{L}\right)/\pi^4\mathrm{EI}n^4$ overlaps with the result in Ref. [10].

4.2. C-F Beam under Sinusoidal Distributed Loads

In the case of C-F beam, the unknown elements of the initial state vector becomes ($w_o=0,\theta_o=0$)

$$M_o=\frac{L^2q_o(\pi\,\mathrm{n}\cos(\pi n)-\sin(\pi n))}{\pi^2n^2}$$
$$T_o=-\frac{L(\cos(\pi n)-1)q_o}{\pi n} \tag{33}$$

The general solution may be written with the help of Eq. (33) as

$$\boldsymbol{S}(x)_{C-F}^{E}=\left\{\begin{matrix}\dfrac{Lq_o\left(\begin{matrix}6L^3\sin\left(\frac{\pi nx}{L}\right)\\+\pi nx\left(\pi nx\left(\begin{matrix}\pi n(x-3L)\cos(\pi n)\\+3L\sin(\pi n)\end{matrix}\right)\right)\\-6L^2\end{matrix}\right)}{6\pi^4\mathrm{EI}n^4}\\-\dfrac{Lq_o\left(\begin{matrix}\pi^2n^2x(x-2L)\cos(\pi n)\\+2L\left(L\left(\cos\left(\frac{\pi nx}{L}\right)-1\right)\right)\\+\pi\,\mathrm{nxsin}(\pi n)\end{matrix}\right)}{2\pi^3\mathrm{EI}n^3}\\\dfrac{Lq_o\left(\begin{matrix}L\left(\sin\left(\frac{\pi nx}{L}\right)-\sin(\pi n)\right)\\+\pi n(L-x)\cos(\pi n)\end{matrix}\right)}{\pi^2n^2}\\\dfrac{Lq_o\left(\cos\left(\frac{\pi nx}{L}\right)-\cos(\pi n)\right)}{\pi n}\end{matrix}\right\} \tag{34}$$

Solutions for the rotation, bending moment and shearing force are found as the same in two beam theories. Some particular values of sectional quantities are

$$w_{L/2}=\frac{L^4q_o\left(\begin{matrix}6\left(\begin{matrix}8\sin\left(\frac{\pi n}{2}\right)\\+\pi n(\pi\,\mathrm{nsin}(\pi n)-4)\end{matrix}\right)\\-5\pi^3n^3\cos(\pi n)\end{matrix}\right)}{48\pi^4\mathrm{EI}n^4},\quad w_L=-\frac{L^4q_o\left(\begin{matrix}2\pi^3n^3\cos(\pi n)\\-3(\pi^2n^2+2)\sin(\pi n)\\+6\pi n\end{matrix}\right)}{6\pi^4\mathrm{EI}n^4}$$

$$\theta_{L/2}=\frac{L^3q_o\left(\begin{matrix}-8\cos\left(\frac{\pi n}{2}\right)\\+\pi n\left(\begin{matrix}3\pi\,\mathrm{ncos}(\pi n)\\-4\sin(\pi n)\end{matrix}\right)+8\end{matrix}\right)}{8\pi^3\mathrm{EI}n^3},\quad \theta_L=\frac{L^3q_o\left(\begin{matrix}(\pi^2n^2-2)\cos(\pi n)\\-2\pi\,\mathrm{nsin}(\pi n)+2\end{matrix}\right)}{2\pi^3\mathrm{EI}n^3} \tag{35}$$

$$M_{L/2}=\frac{L^2q_o\left(\begin{matrix}2\sin\left(\frac{\pi n}{2}\right)\\-2\sin(\pi n)+\pi\,\mathrm{ncos}(\pi n)\end{matrix}\right)}{2\pi^2n^2}$$

$$T_{L/2}=\frac{L(\cos(\frac{\pi n}{2})-\cos(\pi n))q_o}{\pi n},\qquad T_L=0$$

Since $\sin(\pi n)=0$ when n is a positive integer, Eq. (34) may be cast as follows

$$\boldsymbol{S}(x)^{E}_{C-F(n=positive\ integer)} = \left\{ \begin{array}{c} \dfrac{Lq_o\left(\begin{array}{c} 6L^3 \sin\left(\frac{\pi n x}{L}\right) \\ +\pi n x\left(\begin{array}{c}\pi n x(\pi n(x-3L)\cos(\pi n)) \\ -6L^2\end{array}\right)\end{array}\right)}{6\pi^4 \mathrm{EI} n^4} \\ -\dfrac{Lq_o\left(\begin{array}{c}\pi^2 n^2 x(x-2L)\cos(\pi n) \\ +2L^2\left(\cos\left(\frac{\pi n x}{L}\right)-1\right)\end{array}\right)}{2\pi^3 \mathrm{EI} n^3} \\ \dfrac{Lq_o\left(\begin{array}{c} L\sin\left(\frac{\pi n x}{L}\right) \\ +\pi n(L-x)\cos(\pi n)\end{array}\right)}{\pi^2 n^2} \\ \dfrac{Lq_o\left(\cos\left(\frac{\pi n x}{L}\right)-\cos(\pi n)\right)}{\pi n} \end{array} \right\} \tag{36}$$

4.3. C-S Beam under Sinusoidal Distributed Loads

For a fixed-simple supported beam, the elements of the initial state vector are obtained as

$$w_o = 0, \qquad \theta_o = 0$$

$$M_o = \frac{L^2\left((\pi^2 n^2 + 6)\sin(\pi n) - 6\pi n\right)q_o}{2\pi^4 n^4} \tag{37}$$

$$T_o = \frac{L(2\pi n(\pi^2 n^2 + 3) - 3(\pi^2 n^2 + 2)\sin(\pi n))q_o}{2\pi^4 n^4}$$

The elements of the state vector at any section are found as

$$\boldsymbol{S}(x)_{C-S}^{E}=\left\{\begin{matrix}\dfrac{Lq_o\begin{pmatrix}4L^3\sin\left(\frac{\pi nx}{L}\right)\\-2\pi nx(2L^2-3Lx+x^2)\\+x^2\sin(\pi n)\begin{pmatrix}(\pi^2n^2+2)x\\-L(\pi^2n^2+6)\end{pmatrix}\end{pmatrix}}{4\pi^4\mathrm{EI}n^4}\\ \dfrac{Lq_o\begin{pmatrix}2\pi n\begin{pmatrix}-2L^2\cos\left(\frac{\pi nx}{L}\right)\\+2L^2-6Lx+3x^2\end{pmatrix}\\+\mathrm{x}\sin(\pi n)\begin{pmatrix}2L(\pi^2n^2+6)\\-3(\pi^2n^2+2)x\end{pmatrix}\end{pmatrix}}{4\pi^4\mathrm{EI}n^4}\\ \dfrac{Lq_o\begin{pmatrix}\sin(\pi n)\begin{pmatrix}L(\pi^2n^2+6)\\-3(\pi^2n^2+2)x\end{pmatrix}\\+2\pi n\left(\pi\,\mathrm{Ln}\sin\left(\frac{\pi nx}{L}\right)-3L+3x\right)\end{pmatrix}}{2\pi^4n^4}\\ \dfrac{Lq_o\begin{pmatrix}2\pi^3n^3\cos\left(\frac{\pi nx}{L}\right)\\-3(\pi^2n^2+2)\sin(\pi n)\\+6\pi n\end{pmatrix}}{2\pi^4n^4}\end{matrix}\right\} \tag{38}$$

Specific values of the elements of the state vector are

$$w_{L/2}=-\frac{L^4\begin{pmatrix}(\pi^2n^2+10)\sin(\pi n)\\+6\pi n-32\sin\left(\frac{\pi n}{2}\right)\end{pmatrix}q_o}{32\pi^4\mathrm{EI}n^4}$$

$$\theta_{L/2}=\frac{L^3q_o\begin{pmatrix}(\pi^2n^2+18)\sin(\pi n)\\-2\pi n\left(8\cos\left(\frac{\pi n}{2}\right)+1\right)\end{pmatrix}}{16\pi^4\mathrm{EI}n^4},\qquad \theta_L=-\frac{L^3q_o\begin{pmatrix}(\pi^2n^2-6)\sin(\pi n)\\+2\pi n+4\pi\,\mathrm{n}\cos(\pi n)\end{pmatrix}}{4\pi^4\mathrm{EI}n^4}$$

$$M_{L/2}=\frac{L^2\begin{pmatrix}(6-\pi^2n^2)\sin(\pi n)\\+2\pi n\left(2\pi\,\mathrm{n}\sin\left(\frac{\pi n}{2}\right)-3\right)\end{pmatrix}q_o}{4\pi^4n^4} \tag{39}$$

$$T_{L/2}=\frac{Lq_o\begin{pmatrix}2\pi^3n^3\cos\left(\frac{\pi n}{2}\right)\\-3(\pi^2n^2+2)\sin(\pi n)+6\pi n\end{pmatrix}}{2\pi^4n^4}$$

$$T_L=\frac{Lq_o\begin{pmatrix}2\pi^3n^3\cos(\pi n)\\-3(\pi^2n^2+2)\sin(\pi n)+6\pi n\end{pmatrix}}{2\pi^4n^4}$$

In the case of n is a positive integer then the followings are obtained from Eq. (38) ($\sin(\pi n)=0$).

$$\boldsymbol{S}(x)^{E}_{C-S(n=positive\ integer)} = \left\{ \begin{array}{c} \dfrac{Lq_o\begin{pmatrix} 4L^3\sin\left(\frac{\pi n x}{L}\right) \\ -2\pi n x(2L^2-3Lx+x^2) \\ + \end{pmatrix}}{4\pi^4 \mathrm{EI} n^4} \\ \dfrac{Lq_o 2\pi n\begin{pmatrix} -2L^2\cos\left(\frac{\pi n x}{L}\right) \\ +2L^2-6Lx+3x^2 \end{pmatrix}}{4\pi^4 \mathrm{EI} n^4} \\ \dfrac{Lq_o\left(2\pi n\left(\pi\,\mathrm{Lnsin}\left(\frac{\pi n x}{L}\right)-3L+3x\right)\right)}{2\pi^4 n^4} \\ \dfrac{Lq_o\left(2\pi^3 n^3\cos\left(\frac{\pi n x}{L}\right)+6\pi n\right)}{2\pi^4 n^4} \end{array} \right\} \quad (40)$$

4.4. C-S Beam under Sinusoidal Distributed Loads

In this case, the initial bending moment and the initial shearing force are found as ($w_o = 0, \theta_o = 0$)

$$M_o = -\frac{2L^2 q_o(\pi n(\cos(\pi n)+2) - 3\sin(\pi n))}{\pi^4 n^4}$$

$$T_o = \frac{Lq_o\begin{pmatrix} \pi n(\pi^2 n^2 + 6\cos(\pi n) + 6) \\ -12\sin(\pi n) \end{pmatrix}}{\pi^4 n^4} \quad (41)$$

The state vector at any section reads

$$
\boldsymbol{S}(x)^{E}_{C-C} = \left\{ \begin{array}{c}
\dfrac{Lq_o\left(L^3\sin\left(\frac{\pi n x}{L}\right) + x\left(x(2x-3L)\sin(\pi n) - \pi n(x-L)\left(-L + x\cos(\pi n) + x\right)\right)\right)}{\pi^4 \mathrm{EI} n^4} \\
\dfrac{Lq_o\left(\pi L^2(-n)\cos\left(\frac{\pi n x}{L}\right) + (L-x)\left(\pi n(L-3x) + 6x\sin(\pi n)\right) + \pi n x(3x-2L)\cos(\pi n)\right)}{\pi^4 \mathrm{EI} n^4} \\
\dfrac{Lq_o\left(6(L-2x)\sin(\pi n) + \pi n\left(\pi L n\sin\left(\frac{\pi n x}{L}\right) - 4L + 6x\right) - 2\pi n(L-3x)\cos(\pi n)\right)}{\pi^4 n^4} \\
\dfrac{Lq_o\left(\pi n\left(\pi^2 n^2\cos\left(\frac{\pi n x}{L}\right) + 6\cos(\pi n) + 6\right) - 12\sin(\pi n)\right)}{\pi^4 n^4}
\end{array} \right\} \tag{42}
$$

Some values of the elements of the state vector at sections at $x = L/2$ and $x = L$ are

$$
w_{L/2} = \frac{L^4 q_o\left(-\pi n + 8\sin\left(\frac{\pi n}{2}\right) - 4\sin(\pi n) + \pi n\cos(\pi n)\right)}{8\pi^4 \mathrm{EI} n^4}
$$

$$
\theta_{L/2} = -\frac{L^3 q_o\left(\pi n\left(4\cos\left(\frac{\pi n}{2}\right) + \cos(\pi n) + 1\right) - 6\sin(\pi n)\right)}{4\pi^4 \mathrm{EI} n^4}
$$

$$
M_{L/2} = \frac{L^2 q_o\left(\pi n\sin\left(\frac{\pi n}{2}\right) + \cos(\pi n) - 1\right)}{\pi^3 n^3}, \quad M_L = \frac{L^2 q_o\left((\pi^2 n^2 - 6)\sin(\pi n) + 2\pi n + 4\pi n\cos(\pi n)\right)}{\pi^4 n^4} \tag{43}
$$

$$
T_{L/2} = \frac{Lq_o\left(\pi^3 n^3\cos\left(\frac{\pi n}{2}\right) - 12\sin(\pi n) + 6\pi n(\cos(\pi n) + 1)\right)}{\pi^4 n^4}, \quad T_L = \frac{Lq_o\left(\pi(\pi^2 n^2 + 6)n\cos(\pi n) + 6\pi n - 12\sin(\pi n)\right)}{\pi^4 n^4}
$$

When n is chosen as a positive integer ($\sin(\pi n) = 0$) then

$$\boldsymbol{S}(x)^{E}_{C-C(n=positive\ integer)} = \left\{ \begin{array}{c} \dfrac{Lq_o\left(\begin{array}{c} L^3\sin\left(\frac{\pi n x}{L}\right) \\ -x\pi n(x-L)\left(\begin{array}{c} -L \\ +\mathrm{x}\cos(\pi n) \\ +x \end{array}\right) \end{array}\right)}{\pi^4 \mathrm{EI} n^4} \\ \dfrac{Lq_o\left(\begin{array}{c} \pi L^2(-n)\cos\left(\frac{\pi n x}{L}\right) \\ +(L-x)\big(\pi n(L-3x)\big) \\ +\pi n x(3x-2L)\cos(\pi n) \end{array}\right)}{\pi^4 \mathrm{EI} n^4} \\ \dfrac{Lq_o\left(\begin{array}{c} \pi n\left(\begin{array}{c} \pi\,\mathrm{Ln}\sin\left(\frac{\pi n x}{L}\right) \\ -4L+6x \end{array}\right) \\ -2\pi n(L-3x)\cos(\pi n) \end{array}\right)}{\pi^4 n^4} \\ \dfrac{Lq_o\pi n\left(\begin{array}{c} \pi^2 n^2\cos\left(\frac{\pi n x}{L}\right) \\ +6\cos(\pi n)+6 \end{array}\right)}{\pi^4 n^4} \end{array} \right\} \tag{44}$$

5. Solutions for Parabolically Distributed Forces

If a generalized power-type distributed force [10] is concerned (Fig. 7)

$$q(x) = -q_o\left(\frac{x}{L}\right)^n, \qquad n \geq 0 \tag{45}$$

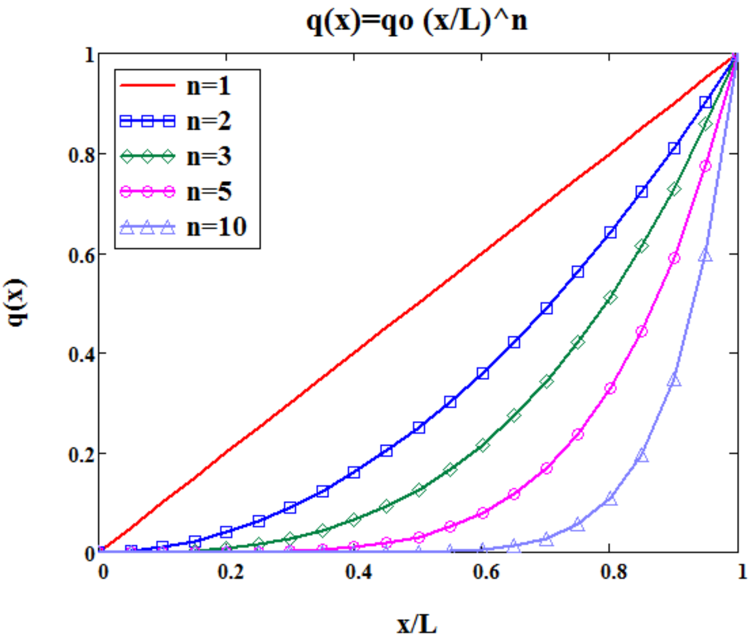


Fig. 7. Generalized power distributed loads

The inhomogeneous solution reads

$$\int_0^x \boldsymbol{F}(x-\xi)\boldsymbol{k}(\xi)\,d\xi = \begin{Bmatrix} \dfrac{L^{-n}x^{n+4}q_o}{\mathrm{EI}(n+1)(n+2)(n+3)(n+4)} \\ -\dfrac{L^{-n}x^{n+3}q_o}{\mathrm{EI}(n^3+6n^2+11n+6)} \\ -\dfrac{L^{-n}x^{n+2}q_o}{n^2+3n+2} \\ -\dfrac{L^{-n}x^{n+1}q_o}{n+1} \end{Bmatrix} \tag{46}$$

5.1. S-S Beam under Parabolically Distributed Loads

In this case the state vector is found as

$$\boldsymbol{S}(x)_{S-S}^{E} = \boldsymbol{F}(x)\boldsymbol{S}(0) + \int_0^x \boldsymbol{F}(x-\xi)\boldsymbol{k}(x)\,d\xi = \begin{Bmatrix} \dfrac{xq_o\begin{pmatrix} L^3(n+1)(n+6) \\ +6x^3\left(\frac{x}{L}\right)^n \\ -L(n+3)(n+4)x^2 \end{pmatrix}}{6\mathrm{EI}(n+1)(n+2)(n+3)(n+4)} \\ \dfrac{q_o\begin{pmatrix} -L^3(n+1)(n+6) \\ -6(n+4)x^3\left(\frac{x}{L}\right)^n \\ +3L(n+3)(n+4)x^2 \end{pmatrix}}{6\mathrm{EI}(n+1)(n+2)(n+3)(n+4)} \\ \dfrac{xq_o\left(L-x\left(\frac{x}{L}\right)^n\right)}{n^2+3n+2} \\ \dfrac{q_o\left(L-(n+2)x\left(\frac{x}{L}\right)^n\right)}{(n+1)(n+2)} \end{Bmatrix} \tag{47}$$

where the initial rotation and the initial shearing force are found as ($w_o = 0, M_o = 0$)

$$\theta_o = -\frac{L^3(n+6)q_o}{6\mathrm{EI}(n+2)(n+3)(n+4)}$$
$$T_o = \frac{Lq_o}{n^2+3n+2} \tag{48}$$

In both theories $\theta(x), M(x)\ and\ T(x)$ are the same for the beam with simply supported at both ends. In Eq. (47) $n = 0$ offers a uniformly distributed force (See Eq. 18).

$$w_{(S-S)(n=0)}^{E} = \frac{x(L^3-2Lx^2+x^3)q_o}{24\mathrm{EI}} = \frac{q_oL^4}{24\mathrm{EI}}\left(\left(\frac{x}{L}\right)-2\left(\frac{x}{L}\right)^3+\left(\frac{x}{L}\right)^4\right)$$
$$\theta_{(S-S)(n=0)}^{E} = -\frac{(L^3-6Lx^2+4x^3)q_o}{24\mathrm{EI}} \tag{49}$$
$$M_{(S-S)(n=0)}^{E} = \frac{1}{2}x(L-x)q_o, \qquad T_{(S-S)(n=0)}^{E} = \frac{1}{2}(L-2x)q_o$$

In the above, $w^E_{(S-S)(n=0)}$ overlaps with the result in Ref. [9]. In Eq. (47) $n = 1$ proposes a linearly distributed force (triangular shape).

$$w^E_{(S-S)(n=1)} = \frac{x\left(14L^3 + \frac{6x^4}{L} - 20Lx^2\right)q_o}{720\text{EI}}$$
$$\theta^E_{(S-S)(n=1)} = \frac{\left(-14L^3 - \frac{30x^4}{L} + 60Lx^2\right)q_o}{720\text{EI}} \quad (50)$$
$$M^E_{(S-S)(n=1)} = \frac{1}{6}x\left(L - \frac{x^2}{L}\right)q_o$$
$$T^E_{(S-S)(n=1)} = \frac{1}{6}\left(L - \frac{3x^2}{L}\right)q_o$$

These formulas coincides with the open literature [3].

$$w(x)_{triangular} = \frac{q_o L^3 x\left(7 + 3\frac{x^4}{L^4} - 10L\frac{x^2}{L^2}\right)}{360\text{EI}}$$
$$\theta_{o-triangular} = \frac{7q_o L^3}{360\text{EI}}, \theta_{L-triangular} = \frac{8q_o L^3}{360\text{EI}} \quad (51)$$
$$M(x)_{triangular} = \frac{q_o Lx}{6}\left(1 - \frac{x^2}{L^2}\right)$$

5.2. C-F Beam under Parabolically Distributed Loads

The non-zero elements of the initial state vector are

$$M_o = -\frac{L^2 q_o}{n+2}, \qquad T_o = \frac{Lq_o}{n+1} \quad (52)$$

Longitudinal variation of the sectional quantities are

$$\boldsymbol{S}(x)^E_{C-F} = \left\{\begin{matrix} \dfrac{x^2 q_o\left(\dfrac{6L^{-n}x^{n+2}}{(n+1)(n+2)(n+3)(n+4)} + \dfrac{3L^2}{n+2} - \dfrac{Lx}{n+1}\right)}{6\text{EI}} \\ \dfrac{xL^{-n}q_o\left((n+3)L^{n+1}\left(\begin{matrix}(n+2)x\\-2L(n+1)\end{matrix}\right) - 2x^{n+2}\right)}{2\text{EI}(n+1)(n+2)(n+3)} \\ \dfrac{L^{-n}q_o\left(L^{n+1}\left(\begin{matrix}(n+2)x\\-L(n+1)\end{matrix}\right) - x^{n+2}\right)}{(n+1)(n+2)} \\ \dfrac{Lq_o - L^{-n}x^{n+1}q_o}{n+1} \end{matrix}\right\} \quad (53)$$

In a cantilever beam $\theta(x), M(x)\ and\ T(x)$ are the same in both beam theories. A uniformly distributed load is obtained with $n = 0$ in Eq. (53) (see Eqn. 23).

$$w^E_{(C-F)(n=0)} = \frac{x^2(6L^2 - 4Lx + x^2)q_o}{24\mathrm{EI}} = \frac{q_o L^4}{24\mathrm{EI}}\left(6\left(\frac{x}{L}\right)^2 - 4\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right)^4\right)$$
$$\theta^E_{(C-F)(n=0)} = -\frac{x(3L^2 - 3Lx + x^2)q_o}{6\mathrm{EI}} \tag{54}$$
$$M^E_{(C-F)(n=0)} = -\frac{1}{2}(L-x)^2 q_o$$
$$T^E_{(C-F)(n=0)} = (L-x)q_o$$

In the above, the deflection formula overlaps by Armagan's [10] expression. Sectional quantities in a cantilever beam under linearly distributed forces is obtained with $n = 1$ in Eq. (53) as follows [3].

$$w^E_{(C-F)(n=1)} = \frac{x^2 q_o}{6\mathrm{EI}}\left(L^2 + \frac{x^3}{20L} - \frac{Lx}{2}\right)$$
$$\theta^E_{(C-F)(n=1)} = -\frac{x(8L^3 - 6L^2x + x^3)q_o}{24\mathrm{EI}L} \tag{55}$$
$$M^E_{(C-F)(n=1)} = -\frac{(L-x)^2(2L+x)q_o}{6L}$$
$$T^E_{(C-F)(n=1)} = \frac{(L-x)(L+x)q_o}{2L}$$

Peddieson et al. [10] presented the following

$$w^E_{(C-F)-Peddieson\ et\ al.[10]} = \frac{L^4 q_o\left(\left(\frac{x}{L}\right)^{4+n} + \frac{(n+1)(n+3)(n+4)}{2} - \frac{(n+2)(n+3)(n+4)}{6}\left(\frac{x}{L}\right)\right)}{\mathrm{EI}(n+1)(n+2)(n+3)(n+4)} \tag{56}$$

However, the transverse displacement in Eq. (53) may be rewritten as

$$w^E_{(C-F)} = \frac{L^4 q_o\left(\left(\frac{x}{L}\right)^{4+n} + \frac{(n+1)(n+3)(n+4)}{2}\left(\frac{x}{L}\right)^2 - \frac{(n+2)(n+3)(n+4)}{6}\left(\frac{x}{L}\right)^3\right)}{\mathrm{EI}(n+1)(n+2)(n+3)(n+4)} \tag{57}$$

The author thinks that there must be typographical errors in Peddieson and et al.'s [10] results.

5.3. C-S Beam under Parabolically Distributed Loads

The bending moment and the shearing force at the initial section are found as ($w_o = 0, \theta_o = 0$)

$$M_o = -\frac{L^2(n+6)q_o}{2(n+2)(n+3)(n+4)}$$
$$T_o = \frac{3L(n+5)q_o}{2(n+1)(n+3)(n+4)} \tag{58}$$

The state vector is to be

$$\boldsymbol{S}(x)^E_{C-S} = \boldsymbol{F}(x)\boldsymbol{S}(0) + \int_0^x \boldsymbol{F}(x-\xi)\boldsymbol{k}(x)\,d\xi = \left\{\begin{array}{c} \dfrac{x^2L^{-n}q_o\left(L^{n+1}\begin{pmatrix} L(n+1)(n+6) \\ -(n+2)(n+5)x \end{pmatrix} \atop +4x^{n+2}\right)}{4\text{EI}(n+1)(n+2)(n+3)(n+4)} \\ \dfrac{xL^{-n}q_o\left(L^{n+1}\begin{pmatrix} 3(n+2)(n+5)x \\ -2L(n+1)(n+6) \end{pmatrix} \atop -4(n+4)x^{n+2}\right)}{4\text{EI}(n+1)(n+2)(n+3)(n+4)} \\ \dfrac{L^{-n}q_o\left(L^{n+1}\begin{pmatrix} 3(n+2)(n+5)x \\ -L(n+1)(n+6) \end{pmatrix} \atop -2(n+3)(n+4)x^{n+2}\right)}{2(n+1)(n+2)(n+3)(n+4)} \\ \dfrac{q_o\left(\dfrac{3L(n+5)}{(n+3)(n+4)} - 2L^{-n}x^{n+1}\right)}{2(n+1)} \end{array}\right\} \tag{59}$$

As stated above, a uniformly distributed force is obtained by substituting $n = 0$ in Eq. (59) (see Eq. 21)

$$w^E_{(C-S)(n=0)} = \frac{x^2(3L-2x)(L-x)q_o}{48\text{EI}}$$
$$\theta^E_{(C-S)(n=0)} = -\frac{x(6L^2-15Lx+8x^2)q_o}{48\text{EI}}$$
$$M^E_{(C-S)(n=0)} = -\frac{1}{8}(L-4x)(L-x)q_o \tag{60}$$
$$T^E_{(C-S)(n=0)} = \frac{1}{8}(5L-8x)q_o$$

To get a linearly distributed force, $n = 1$ should be used in Eq. (59).

$$w^E_{(C-S)(n=1)} = \frac{x^2(7L^3-9L^2x+2x^3)q_o}{240\text{EI}L}$$
$$\theta^E_{(C-S)(n=1)} = -\frac{x(14L^3-27L^2x+10x^3)q_o}{240\text{EI}L}$$
$$M^E_{(C-S)(n=1)} = -\frac{(7L^3-27L^2x+20x^3)q_o}{120L} \tag{61}$$
$$T^E_{(C-S)(n=1)} = \frac{1}{4}\left(\frac{9L}{10}-\frac{2x^2}{L}\right)q_o$$

5.4. C- C Beam under Parabolically Distributed Loads

Initial bending moment and shearing force at the initial section of a C-C Euler-Bernoulli beam are obtained as ($w_o = 0, \theta_o = 0$)

$$M_o = -\frac{2L^2 q_o}{n^3 + 9n^2 + 26n + 24}$$
$$T_o = \frac{6Lq_o}{n^3 + 8n^2 + 19n + 12} \tag{62}$$

The state vector at any section is

$$\boldsymbol{S}(x)^E_{C-C} = \boldsymbol{F}(x)\boldsymbol{S}(0) + \int_0^x \boldsymbol{F}(x-\xi)\boldsymbol{k}(x)\,d\xi = \begin{Bmatrix} \dfrac{x^2 L^{-n} q_o \left(L^{n+1} \begin{pmatrix} L(n+1) \\ -(n+2)x \end{pmatrix} + x^{n+2} \right)}{\mathrm{EI}(n+1)(n+2)(n+3)(n+4)} \\ \dfrac{x L^{-n} q_o \left(L^{n+1} \begin{pmatrix} 3(n+2)x \\ -2L(n+1) \end{pmatrix} - (n+4)x^{n+2} \right)}{\mathrm{EI}(n+1)(n+2)(n+3)(n+4)} \\ \dfrac{L^{-n} q_o \left(-2L^{n+1} \begin{pmatrix} L(n+1) \\ -3(n+2)x \end{pmatrix} - (n+3)(n+4)x^{n+2} \right)}{(n+1)(n+2)(n+3)(n+4)} \\ q_o \left(\dfrac{6L}{n^3 + 8n^2 + 19n + 12} - \dfrac{L^{-n} x^{n+1}}{n+1} \right) \end{Bmatrix} \tag{63}$$

In the above $n = 0$ means a uniformly distributed force as follows (see Eq. 25)

$$w^E_{(C-C)(n=0)} = \frac{x^2 (L-x)^2 q_o}{24\mathrm{EI}}$$
$$\theta^E_{(C-C)(n=0)} = -\frac{x(L-2x)(L-x)q_o}{12\mathrm{EI}}$$
$$M^E_{(C-C)(n=0)} = -\frac{1}{12}(L^2 - 6Lx + 6x^2)q_o$$
$$T^E_{(C-C)(n=0)} = \frac{1}{2}(L-2x)q_o \tag{64}$$

For $n = 1$ the following is achieved

$$w^E_{(C-C)(n=1)} = \frac{x^2(L-x)^2(2L+x)q_o}{120\mathrm{EI}L}, \quad \theta^E_{(C-C)(n=1)} = -\frac{x(4L^3 - 9L^2 x + 5x^3)q_o}{120\mathrm{EI}L}$$
$$M^E_{(C-C)(n=1)} = -\frac{(2L^3 - 9L^2 x + 10x^3)q_o}{60L}, \quad T^E_{(C-C)(n=1)} = \left(\frac{3L}{20} - \frac{x^2}{2L}\right) q_o \tag{65}$$

It may be noted that, when $n = 0$, $\theta(x), M(x)$, and $T(x)$ are all the same in both two beam theories. However, they are no longer the same for different values of n.

6. Solutions for Concentrated Force and Moments

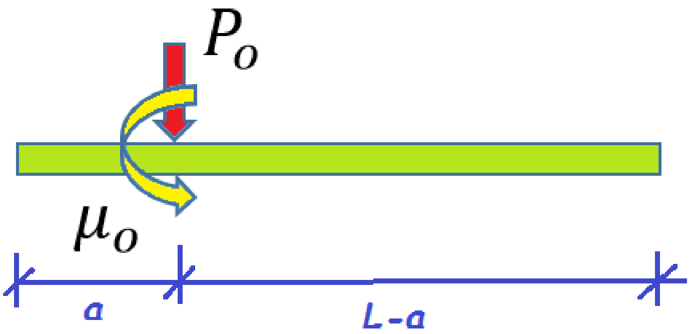


Fig. 8. Concentrated force and couple acting at section $x = a$

With the help of a discontinuity matrix due to a single couple moment, μ_o, and a single force, P_o, at section $x = a$ in Eq. (11), $\boldsymbol{K}(a) = \{0 \quad 0 \quad -\mu_o \quad -P_o\}^T$, the following may be obtained for $a \leq x \leq L$ (Fig. 8)

$$\boldsymbol{F}(x-a)\boldsymbol{K}(a) = \begin{Bmatrix} \dfrac{(a-x)^2\left((x-a)P_o + 3\mu_o\right)}{6\text{EI}} \\ \dfrac{(a-x)\left((x-a)P_o + 2\mu_o\right)}{2\text{EI}} \\ (a-x)P_o - \mu_o \\ -P_o \end{Bmatrix} \qquad (66)$$

When only concentrated loads are considered, the general solution is defined in two regions, which are defined as before and after $x = a$, as follows (see Eq. 14)

$$\begin{gathered} \boldsymbol{S}(x)^{I} = \boldsymbol{F}(x)\boldsymbol{S}(0) \\ \boldsymbol{S}(x)^{II} = \boldsymbol{F}(x)\boldsymbol{S}(0) + \boldsymbol{F}(x-a)\boldsymbol{K}(a) \end{gathered} \qquad (67)$$

6.1. S-S Beam under Concentrated Force and Moments

In this case the non-zero elements of $\boldsymbol{S}(0)$ are

$$\begin{gathered} \theta_o = \frac{(3a^2 - 6aL + 2L^2)\mu_o - a(a-2L)(a-L)P_o}{6\text{EI}L} \\ T_o = \frac{(L-a)P_o + \mu_o}{L} \end{gathered} \qquad (68)$$

The elements of the state vector are

$$\boldsymbol{S}(x)^{I}_{S-S}=\boldsymbol{F}(x)\boldsymbol{S}(0)=\begin{Bmatrix} \dfrac{\begin{pmatrix} x(a-L)P_o(a^2-2aL+x^2) \\ -x\mu_o\begin{pmatrix}3a^2-6aL\\+2L^2+x^2\end{pmatrix}\end{pmatrix}}{6\mathrm{EI}L} \\ \dfrac{\begin{pmatrix} \mu_o(3a^2-6aL+2L^2+3x^2) \\ -(a-L)P_o\begin{pmatrix}a^2-2aL\\+3x^2\end{pmatrix}\end{pmatrix}}{6\mathrm{EI}L} \\ \dfrac{x\big((L-a)P_o+\mu_o\big)}{L} \\ \dfrac{(L-a)P_o+\mu_o}{L} \end{Bmatrix} \tag{69}$$

$$\boldsymbol{S}(x)^{II}_{S-S}=\boldsymbol{F}(x)\boldsymbol{S}(0)+\boldsymbol{F}(x-a)\boldsymbol{K}(a)=\begin{Bmatrix} \dfrac{\left((L-x)\begin{pmatrix}\mu_o\big(3a^2+x(x-2L)\big)\\-aP_o\big(a^2+x(x-2L)\big)\end{pmatrix}\right)}{6\mathrm{EI}L} \\ \dfrac{\begin{pmatrix}\mu_o(3a^2+2L^2-6Lx+3x^2)\\-aP_o(a^2+2L^2-6Lx+3x^2)\end{pmatrix}}{6\mathrm{EI}L} \\ \dfrac{(L-x)(aP_o-\mu_o)}{L} \\ \dfrac{\mu_o-aP_o}{L} \end{Bmatrix}$$

When the beam is only subjected to a single force at the mid-span ($\mu_o=0, a=L/2$) then Eq. (69) becomes

$$\boldsymbol{S}(x)^{I}_{S-S}=\begin{Bmatrix} \dfrac{(3L^2x-4x^3)P_o}{48\mathrm{EI}} \\ -\dfrac{(L^2-4x^2)P_o}{16\mathrm{EI}} \\ \dfrac{xP_o}{2} \\ \dfrac{P_o}{2} \end{Bmatrix},\quad \boldsymbol{S}(x)^{II}_{S-S}=\begin{Bmatrix} -\dfrac{(L-x)(L^2-8Lx+4x^2)P_o}{48\mathrm{EI}} \\ -\dfrac{(L-2x)(3L-2x)P_o}{16\mathrm{EI}} \\ \dfrac{1}{2}(L-x)P_o \\ -\dfrac{P_o}{2} \end{Bmatrix} \tag{70}$$

The dimensionless displacement may be defined as $\bar{w}=EIw/(P_oL^3)$ for a point force. From Eq. (70) with $x=L/2$ it is found as $\bar{w}_{max}=1/48=0.020833$. Aydoğdu [12] reported it as $\bar{w}_{max}=0.022222$.

6.2. C-C Beam under Concentrated Force and Moments

The unknown elements of the initial state vector are

$$M_o=-\frac{(a-L)\big((L-3a)\mu_o+a(a-L)P_o\big)}{L^2},\quad T_o=\frac{(a-L)((a-L)(2a+L)P_o-6a\mu_o)}{L^3} \tag{71}$$

With the help of the above, the following is written

$$\boldsymbol{S}(x)^{I}_{C-C}=\boldsymbol{F}(x)\boldsymbol{S}(0)=\begin{Bmatrix} \dfrac{x^2(a-L)\left(\begin{array}{c}3\mu_o(-3aL+2ax+L^2)\\ +(a-L)P_o\left(\begin{array}{c}3aL\\ -x(2a+L)\end{array}\right)\end{array}\right)}{6\mathrm{EI}L^3} \\ -\dfrac{x(a-L)\left(\begin{array}{c}2\mu_o(3a(x-L)+L^2)\\ +(a-L)P_o\left(\begin{array}{c}2aL\\ -x(2a+L)\end{array}\right)\end{array}\right)}{2\mathrm{EI}L^3} \\ -\dfrac{(a-L)\left(\begin{array}{c}\mu_o(-3aL+6ax+L^2)\\ +(a-L)P_o\left(\begin{array}{c}aL\\ -x(2a+L)\end{array}\right)\end{array}\right)}{L^3} \\ \dfrac{(a-L)\big((a-L)(2a+L)P_o-6a\mu_o\big)}{L^3} \end{Bmatrix} \tag{72}$$

$$\boldsymbol{S}(x)^{II}_{C-C}=\boldsymbol{F}(x)\boldsymbol{S}(0)+\boldsymbol{F}(x-a)\boldsymbol{K}(a)=\begin{Bmatrix} \dfrac{a(L-x)^2\left(\begin{array}{c}3\mu_o(a(L+2x)-2Lx)\\ -aP_o(a(L+2x)-3Lx)\end{array}\right)}{6\mathrm{EI}L^3} \\ -\dfrac{a(L-x)\left(\begin{array}{c}aP_o(2ax+L^2-3Lx)\\ -2\mu_o(3ax+L^2-3Lx)\end{array}\right)}{2\mathrm{EI}L^3} \\ \dfrac{a\left(\begin{array}{c}aP_o(-aL+2ax+2L^2-3Lx)\\ +\mu_o(3a(L-2x)-4L^2+6Lx)\end{array}\right)}{L^3} \\ \dfrac{a(6(L-a)\mu_o+a(2a-3L)P_o)}{L^3} \end{Bmatrix}$$

When C-C Euler beam is only subjected to a single force at the mid-span ($\mu_o=0, a=L/2$) then Eq. (72) becomes

$$\boldsymbol{S}(x)^{I}_{C-C}=\begin{Bmatrix} \dfrac{x^2(3L-4x)P_o}{48\mathrm{EI}} \\ -\dfrac{x(L-2x)P_o}{8\mathrm{EI}} \\ -\dfrac{1}{8}(L-4x)P_o \\ \dfrac{P_o}{2} \end{Bmatrix}, \qquad \boldsymbol{S}(x)^{II}_{C-C}=\begin{Bmatrix} -\dfrac{(L-4x)(L-x)^2P_o}{48\mathrm{EI}} \\ -\dfrac{(L-2x)(L-x)P_o}{8\mathrm{EI}} \\ \dfrac{1}{8}(3L-4x)P_o \\ -\dfrac{P_o}{2} \end{Bmatrix} \tag{73}$$

6.3. C-F Beam under Concentrated Force and Moments

For a C-F Euler-Bernoulli beam under concentrated force and moments, the initial state vector is determined as

$$\boldsymbol{S}(0)=\{0 \quad 0 \quad (\mu_o-aP_o) \quad P_o\}^T \tag{74}$$

Variation of the sectional quantities along the beam is

$$\boldsymbol{S}(x)_{C-F}^{I}=\boldsymbol{F}(x)\boldsymbol{S}(0)=\begin{Bmatrix} -\dfrac{x^2((x-3a)P_o+3\mu_o)}{6\mathrm{EI}} \\ \dfrac{x((x-2a)P_o+2\mu_o)}{2\mathrm{EI}} \\ (x-a)P_o+\mu_o \\ P_o \end{Bmatrix} \tag{75}$$

$$\boldsymbol{S}(x)_{C-F}^{II}=\boldsymbol{F}(x)\boldsymbol{S}(0)+\boldsymbol{F}(x-a)\boldsymbol{K}(a)=\begin{Bmatrix} \dfrac{\begin{pmatrix} 3a(a-2L)\mu_o \\ -(a^3-3a^2L+3aL^2-3L^3)P_o \end{pmatrix}}{6\mathrm{EI}} \\ \dfrac{2a\mu_o-(a^2-2aL+2L^2)P_o}{2\mathrm{EI}} \\ (a-L)P_o \\ 0 \end{Bmatrix}$$

If only a single force acts on a C-F Euler-Bernoulli beam at section $x=L$, solutions in two regions become ($\mu_o=0\ and\ \ a=L$)

$$\boldsymbol{S}(x)_{C-F}^{I}=\begin{Bmatrix} \dfrac{x^2(3L-x)P_o}{6\mathrm{EI}} \\ \dfrac{x(x-2L)P_o}{2\mathrm{EI}} \\ (x-L)P_o \\ P_o \end{Bmatrix} \tag{76}$$

$$\boldsymbol{S}(x)_{C-F}^{II}=\begin{Bmatrix} \dfrac{L^3P_o}{3\mathrm{EI}} \\ -\dfrac{L^2P_o}{2\mathrm{EI}} \\ 0 \\ 0 \end{Bmatrix}$$

6.4. C-S Beam under Concentrated Force and Moments

Unknown elements of the initial state vector are

$$M_o=\frac{(3a^2-6aL+2L^2)\mu_o-a(a-2L)(a-L)P_o}{2L^2} \tag{77}$$

$$T_o=\frac{(a^3-3a^2L+2L^3)P_o-3a(a-2L)\mu_o}{2L^3}$$

For this boundary condition, the whole elements of the state vector at any section are defined by

$$\boldsymbol{S}(x)_{C-S}^{I}=\boldsymbol{F}(x)\boldsymbol{S}(0)=\left\{\begin{matrix}\dfrac{x^2\begin{pmatrix}3\mu_o\begin{pmatrix}a^2(x-3L)\\+2aL(3L-x)-2L^3\end{pmatrix}\\+(a-L)P_o\begin{pmatrix}a^2(3L-x)\\+2aL(x-3L)+2L^2x\end{pmatrix}\end{pmatrix}}{12\mathrm{EI}L^3}\\ \dfrac{\begin{pmatrix}x\mu_o\begin{pmatrix}a^2(6L-3x)\\+6aL(x-2L)+4L^3\end{pmatrix}\\+x(a-L)P_o\begin{pmatrix}x(a^2-2aL-2L^2)\\-2aL(a-2L)\end{pmatrix}\end{pmatrix}}{4\mathrm{EI}L^3}\\ \dfrac{\begin{pmatrix}\mu_o\begin{pmatrix}3a^2(L-x)\\+6aL(x-L)+2L^3\end{pmatrix}\\-(a-L)P_o\begin{pmatrix}a^2(L-x)\\+2aL(x-L)+2L^2x\end{pmatrix}\end{pmatrix}}{2L^3}\\ \dfrac{\begin{pmatrix}(a^3-3a^2L+2L^3)P_o\\-3a(a-2L)\mu_o\end{pmatrix}}{2L^3}\end{matrix}\right\} \tag{78}$$

$$\boldsymbol{S}(x)_{C-S}^{II}=\boldsymbol{F}(x)\boldsymbol{S}(0)+\boldsymbol{F}(x-a)\boldsymbol{K}(a)=\left\{\begin{matrix}\dfrac{a(L-x)\begin{pmatrix}aP_o\begin{pmatrix}-2aL^2+x^2(a-3L)\\-2Lx(a-3L)\end{pmatrix}\\+3\mu_o\begin{pmatrix}a(2L^2+2Lx-x^2)\\+2Lx(x-2L)\end{pmatrix}\end{pmatrix}}{12\mathrm{EI}L^3}\\ \dfrac{a\begin{pmatrix}aP_o\begin{pmatrix}-Lx(2a+3x)\\+ax^2-2L^3+6L^2x\end{pmatrix}\\+\mu_o\begin{pmatrix}6Lx(a+x)-3ax^2\\+4L^3-12L^2x\end{pmatrix}\end{pmatrix}}{4\mathrm{EI}L^3}\\ \dfrac{a(L-x)(3(a-2L)\mu_o-a(a-3L)P_o)}{2L^3}\\ \dfrac{a(a(a-3L)P_o-3(a-2L)\mu_o)}{2L^3}\end{matrix}\right\}$$

If $\mu_o=0$ and $a=L/2$, Eq. (77) turns to be

$$\boldsymbol{S}(x)_{C-S}^{I}=\left\{\begin{matrix}\dfrac{x^2(9L-11x)P_o}{96\mathrm{EI}}\\ \dfrac{x(11x-6L)P_o}{32\mathrm{EI}}\\ \dfrac{1}{16}(11x-3L)P_o\\ \dfrac{11P_o}{16}\end{matrix}\right\},\quad \boldsymbol{S}(x)_{C-S}^{II}=\left\{\begin{matrix}-\dfrac{(L-x)(2L^2-10Lx+5x^2)P_o}{96\mathrm{EI}}\\ -\dfrac{(4L^2-10Lx+5x^2)P_o}{32\mathrm{EI}}\\ \dfrac{5}{16}(L-x)P_o\\ -\dfrac{5P_o}{16}\end{matrix}\right\} \tag{79}$$

7. Conclusions

In the present study some remarkable formulas are proposed for the bending behavior of classically supported Euler-Bernoulli beams under both distributed and concentrated loads via the transfer matrix approach. For classical boundary conditions it is observed that, Euler-Bernoulli beam solutions are independent from L/h ratios. This is an expected conclusion. The present formulas which also comprise the point and distributed couple moments may be very useful to the readers. It is worth noting that sectional quantities at positive sections may be obtained by using those formulas. Since the present analysis is a linear elastic, the superposition principle is hold under combined loads.

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