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Exact Radial Natural Frequencies of Functionally Graded Hollow Long Cylinders

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Abstract

In this work the exact free axisymmetric pure radial vibration of hollow infinite cylinders made of hypothetically functionally power-graded materials having identical inhomogeneity indexes for both Young's modulus and the material density is addressed. The equation of motion is obtained as a linear second-order Bessel's ordinary differential equation with constant coefficients based on the axisymmetric linear elasticity theory. For traction free boundaries, a closed form frequency equation is offered. After verifying the present results for cylinders made of both isotropic and homogeneous materials, and isotropic functionally graded materials, an extensive parametric study is carried out to investigate the influences of both the thickness and inhomogeneity indexes on the natural frequencies. Results are presented in both graphical and tabular forms. It was revealed that the fundamental frequency in the radial mode is principally affected from the inhomogeneity parameters than the higher ones. However, the natural frequencies except the fundamental ones are dramatically affected from the thickness of the cylinder. As the thickness decreases, the natural frequencies considerably increase. It is also revealed that, there is a linear relationship between the fundamental frequency and others in higher modes.

Keywords: Free vibration, natural frequency, thick-walled hollow cylinder, functionally graded.

1. Introduction

Vibration of thin/thick-walled cylinders is of great significance in many engineering applications such as pressure vessel, heat exchangers, nuclear reactor containments, various pipes and tubes. Pioneering studies concerning the vibration of cylinders date back late of 1800s. One of the earliest works on the vibration of cylinders was carried out by Chree [1]. Using the linear three dimensional elasticity theory, Greenspon [2] studied the flexural vibrations of infinitely long traction free hollow thick-walled cylinders. Gazis [3] studied the vibration of infinitely long traction free hollow cylinders on the basis of three dimensional elasticity theory. Gladwell and Tahbildar [4] solved the problem of axisymmetric vibrations of cylinders with the help of the finite-element method. Gladwell and Vijay [5] also analyzed the vibration of free finite length circular cylinders based on the finite element approach. Hutchinson [6] first handled the vibrations of finite length rods and solid cylinders on the basis of linear 3D elasticity and offered a semi-analytical highly accurate method to solve the problem. Then, Hutchinson and El-Azhari [7] applied the same method for the vibrations of free hollow finite length circular cylinders. By employing the energy method based on the 3D theory of elasticity, Singal and Williams [8] studied theoretically and experimentally the vibrations of thick hollow cylinders and rings. The axisymmetric stress-free vibration of a thick elastic cylinder has been studied under plane strain conditions by Gosh [9]. Gosh [9] obtained the solution for forced vibration by using the Laplace transform and presented natural frequency and dynamic stresses for various types of loading, Poisson's ratio and aspect ratios of the cylinder. Leissa and So [10], and So and Leissa [11] studied three-dimensional analysis of the vibrations of free and cantilevered solid cylinders using simple algebraic polynomials in the Ritz method. Liew et al. [12] investigated the free vibrations of stress free hollow cylinders of arbitrary cross-section based on the three dimensional energy displacement expressions. Hung et al. [13] considered the free vibration of cantilevered cylinders. Wang and Williams [14] presented vibrational modes of thick-walled cylinders of finite length based on the finite element method. On the basis of linear 3D theory of elasticity, and by using the Ritz method and Chebyshev polynomials, Zhou et al. [15] worked on the vibration analysis of solid and hollow circular cylinders including rods and curved panels. In this general semi-analytical series solution having high accuracy and good convergence, offered by Zhou et al. [15], the technique of variables separation is developed for various boundary conditions. Mofakhami et al. [16] studied the free vibration of cylinders with finite length under fixed-fixed and free-free boundaries based on the solutions of infinite cylinders and the technique of separation of variables. Abbas [17] treated with the free vibration of a poroelastic hollow cylinder. Yahya and Abd-Alla [18] considered pure radial vibrations in an isotropic elastic hollow cylinder with rotation.

As time progresses and engineers familiarize themselves with new advanced materials such as anisotropic, functionally graded, carbon nanotube composites, studies have focused on the vibration problems of cylinders made of such advanced materials. From those Nelson et al. [19] worked on vibration and waves in laminated orthotropic circular cylinders. Vibration of anisotropic composite cylinders is addresses by Huang and Dong [20]. Yuan and Hsih [21] investigated three dimensional wave propagation in composite cylindrical shells. By using the Ritz method, Kharouf and Heyliger [22] presented a numerical method for finding approximate solutions to static and axisymmetric vibration problems for piezoelectric cylinders, including those composed of more than one material. Markus and Mead [23-24] studied both axisymmetric and asymmetric wave motion in orthotropic cylinders. Ding et al. [25-26] studied elasto-dynamic and thermoelastic-dynamic problems of a non-homogeneous orthotropic hollow cylinders.

As to the functionally graded materials (FGM), Heyliger and Jilani [27] studied the free vibrations of inhomogeneous elastic cylinders and spheres. By using strains-displacement relations from Love’s shell theory and the eigenvalue governing equation from Rayleigh-Ritz method, Loy et al. [28] presented a study on the vibration of cylindrical shells made of a functionally graded material (FGM) composed of stainless steel and nickel. The properties are graded in the thickness direction according to a volume fraction power-law distribution in Loy et al’s [28] study. Their results showed that the frequency characteristics are similar to that observed for homogeneous isotropic cylindrical shells and the frequencies are affected by the constituent volume fractions and the configurations of the constituent materials. Han et al. [29] presented an analytical-numerical method for analyzing characteristics of waves in a cylinder composed of functionally graded material (FGM) by dividing the FGM cylinder into a number of annular elements with three-nodal-lines in the wall thickness. Han et al. [29] assumed a linear variation of material properties along the thickness direction and used the Hamilton principle to develop the dispersion equations for the cylinder. Their numerical results demonstrated that the ratio of radius to thickness has a stronger influence on the frequency spectra in the circumferential wave than on that in the axial wave. Patel et al. [30] studied the free vibration analysis of functionally graded elliptical cylindrical shells based on the higher-order theory. Pelletier and Vel [31] studied analytically the steady-state thermoelastic response of functionally graded orthotropic cylindrical shells. Arciniega and Reddy [32] considered a large deformation analysis of functionally graded shells. Yang and Shen [33] investigated the free vibration of shear deformable functionally graded cylindrical panels. Jianqiao et al. [34] dealt with wave propagation in non-homogeneous magneto-electro-elastic hollow cylinders. Abd-Alla et al. [35] studied influences of the inhomogeneity on the composite infinite cylinder of isotropic material. Based on the first-order shear deformation theory and linear elasticity, Tornabene et al. [36] studied the dynamic behavior of functionally graded moderately thick conical, cylindrical shells and annular plates via the generalized differential quadrature (GDQ) method. They considered two different power-law distributions for the ceramic volume fraction. Keleş and Tutuncu [37] analytically performed free and forced vibration analyses of power-law graded hollow cylinders and spheres. Although their subject matters are out of the present study, it may be useful to cite some studies concerning the vibration of cylinders and cylindrical shells which are functionally graded with state-of-the-art technological structural materials [38-40] in the open literature.

As seen from the above literature survey, vibration problems of solid/hollow cylinders made of functionally graded materials are not studied widely over the time. This was the motivation to the author. As a fundamental work, the present study may be thought of as an extension of Gosh's [9] study to traction free cylinders made of isotropic functionally graded materials. In the present study, to achieve an analytical solution for natural frequencies in purely radial modes, the inhomogeneity indexes of both elasticity modulus and Poisson's ratio are assumed to be identical by necessity. As stated in the related section, otherwise, it is not possible to get a closed form solution. In these cases, the employment of any numerical procedure becomes compulsory. One of the aims of the present study is to have a rough idea about the general response of such cylinders to the free vibration before numerically treatment of the problem. In the parametric study, almost all the variables which substantially influence on the natural frequencies are considered except the effect of Poisson's ratio. In Gosh's study [9] the effect of Poisson's ratios on the natural frequencies was clearly disclosed. The numerical results given here may also be served as a benchmark solution to some advanced numerical studies.

2. Clarification of the Problem

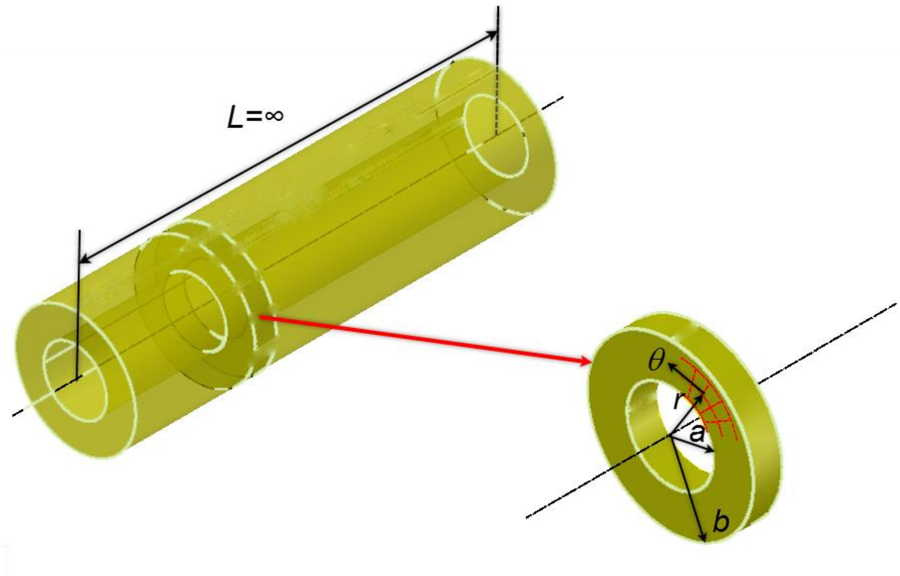

Fig. 1. Geometry of a cylinder

Let's consider a thick-walled hollow cylinder with inner radius a and outer radius b (Fig. 1). Under axisymmetric conditions, relationships between strain and displacement are written in polar coordinates as

$$\varepsilon_r = \frac{du_r}{dr} \quad ; \quad \varepsilon_\theta = \frac{u_r}{r} \tag{1}$$

where the radial unit strain and the circumferential unit strain are symbolized by ε_r , and ε_θ, respectively. In Eq. (1), the radial displacement is represented by u_r. If σ_r , and σ_θ signify the radial stress and the hoop stress, respectively, then Hooke's law for an infinite cylinder made of an isotropic and homogeneous linear elastic material is given by

$$\begin{aligned} \sigma_r &= C_{11}(r)\,\varepsilon_r + C_{12}(r)\,\varepsilon_\theta \\ \sigma_\theta &= C_{12}(r)\,\varepsilon_r + C_{11}(r)\,\varepsilon_\theta \end{aligned} \tag{2}$$

Where under plain strain assumptions

$$\begin{aligned} C_{11}(r) &= E(r)\frac{(1-\nu)}{(1+\nu)(1-2\nu)} \\ C_{12}(r) &= E(r)\frac{\nu}{(1+\nu)(1-2\nu)} \end{aligned} \tag{3}$$

For an isotropic but non-homogeneous FGM material, the material grading rule in the radial direction is assumed to obey the following simple power rule

$$\begin{aligned} E(r) &= E_b\left(\frac{r}{b}\right)^{\eta} \\ \rho(r) &= \rho_b\left(\frac{r}{b}\right)^{\eta} \end{aligned} \tag{4}$$

where ρ is the material density. The material inhomogeneity index is denoted by η; E_b and ρ_b are the reference values of the mixture of the material at the outer surface. Poisson's ratios of both graded materials are assumed to be unaffected along the radial direction as in the most of the related realm.

In Eq. (4) those properties do not completely correspond to a physical material since both Young's modulus and density are assumed to have the same inhomogeneity index. To get an analytical solution to the problem, as seen later, taking both inhomogeneity indexes as if they are identical is going to be inevitable because it is not possible to find a closed-form solution in other choices which require exactly numerical solution techniques.

If the body forces are neglected, the equation of motion in the radial direction is written as follows

$$\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_r) - \frac{\sigma_\theta}{r} = \rho\frac{\partial^2 u_r}{\partial t^2} \tag{5}$$

where t is the time. Substitution both Eq. (1) and (2) together with Eq. (4) into Eq. (5) gives

$$\frac{\partial^2 u_r}{\partial r^2} + \left(\frac{1+\eta}{r}\right)\frac{\partial u_r}{\partial r} - \frac{\left(1-\eta\dfrac{C_{12}(r)}{C_{11}(r)}\right)}{r^2}u_r = -\left(\frac{\rho(r)}{C_{11}(r)}\right)\frac{\partial^2 u_r}{\partial t^2} \tag{6}$$

By assuming a harmonic motion, $u_r(r,t) = u_r^*(r)\,e^{i\omega t}$, with an angular velocity $\omega\,(rad/s)$, Eq. (6) gives way to Bessel's differential equations [41-43].

$$-\left(\frac{1-\eta\left(\frac{\nu}{1-\nu}\right)}{r^2}+\Omega^2\right)u_r^*+\left(\frac{1+\eta}{r}\right)\frac{du_r^*}{dr}+\frac{d^2u_r^*}{dr^2}=0 \tag{7}$$

where

$$\Omega=\sqrt{\frac{\rho(r)}{C_{11}(r)}}\omega=\sqrt{\frac{(1-2\nu)(1+\nu)\rho_b}{(1-\nu)E_b}}\omega=\sqrt{\frac{(1-\nu-2\nu^2)\rho_b}{(1-\nu)E_b}}\omega=\frac{\beta}{a} \tag{8}$$

and β is the dimensionless natural frequency. The solution to this equation, Eq. (7), is going to be in the form of [41-43]

$$u_r^*(\mathrm{r})=r^{-\eta/2}\left(C_1 J_{\frac{\xi}{2}}(r\Omega)+C_2 Y_{\frac{\xi}{2}}(r\Omega)\right) \tag{9}$$

where C_1 and C_2 are arbitrary constants and $J_{\frac{\xi}{2}}(r\Omega)$ and $Y_{\frac{\xi}{2}}(r\Omega)$ denote Bessel's functions of the first and second kind of order $\frac{\xi}{2}$, respectively, and ξ is

$$\xi=\sqrt{4+\eta^2-4\eta\left(\frac{\nu}{1-\nu}\right)}=\sqrt{\frac{(-4-\eta^2+(2+\eta)^2\nu)}{(\nu-1)}} \tag{10}$$

The first derivative of the solution of the radial displacement, u_r^*, and the radial stress, σ_r^*, may be obtained in terms of integration constants, C_1 and C_2, as follows

$$\begin{aligned}\frac{du_r^*}{dr}&=r^{-\eta/2}\left(\frac{1}{2}C_1\Omega\left(J_{\frac{\xi}{2}-1}(r\Omega)-J_{\frac{\xi}{2}+1}(r\Omega)\right)+C_2\Omega\left(Y_{\frac{\xi}{2}-1}(r\Omega)-Y_{\frac{\xi}{2}+1}(r\Omega)\right)\right)\\&-\frac{1}{2}\eta r^{-\frac{\eta}{2}-1}\left(C_1 J_{\frac{\xi}{2}}(r\Omega)+C_2 Y_{\frac{\xi}{2}}(r\Omega)\right)\\ \sigma_r^*(r)&=\frac{E_b r^{-\frac{\eta}{2}-1}\left(\frac{r}{b}\right)^\eta}{2(2\nu^2+\nu-1)}\left(C_1(\eta-(\eta+2)\nu)J_{\frac{\xi}{2}}(r\Omega)+C_1(\nu-1)r\Omega J_{\frac{\xi-2}{2}}(r\Omega)-C_1(\nu-1)r\Omega J_{\frac{\xi+2}{2}}(r\Omega)\right.\\&\left.+C_2(\eta-(\eta+2)\nu)Y_{\frac{\xi}{2}}(r\Omega)+C_2(\nu-1)r\Omega Y_{\frac{\xi-2}{2}}(r\Omega)-C_2(\nu-1)r\Omega Y_{\frac{\xi+2}{2}}(r\Omega)\right)\end{aligned} \tag{11}$$

Eqs. (9) and (11) are used when applying the boundary conditions given at both surfaces. The radial displacement vanishes, $u_r^*=0$, at the fixed surface while the radial stress becomes zero, $\sigma_r^*=0$, at the free surface. For instance, if surface-free boundaries are considered then one may obtain

$$\begin{Bmatrix}\sigma_r^*(a)\\ \sigma_r^*(b)\end{Bmatrix}=\begin{bmatrix}A_{1,1} & A_{1,2}\\ A_{2,1} & A_{2,2}\end{bmatrix}\begin{Bmatrix}C_1\\ C_2\end{Bmatrix}=\boldsymbol{A}\begin{Bmatrix}C_1\\ C_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix} \tag{12}$$

To get non-trivial solutions, natural frequencies are determined from the frequencies which make the determinant of the characteristic coefficient matrix, $\boldsymbol{A}$, zero. Considering traction-free boundary conditions for a hollow infinite cylinder made of a hypothetically functionally power-graded material, elements of the coefficient matrices are found in closed forms as follows

$$A_{1,1}=\frac{a^{-\frac{\eta}{2}-1}\left(\frac{a}{b}\right)^{\eta}\left(a(\nu-1)\Omega J_{\frac{\xi-2}{2}}(a\Omega)+(\eta-(\eta+2)\nu)J_{\frac{\xi}{2}}(a\Omega)-a(\nu-1)\Omega J_{\frac{\xi+2}{2}}(a\Omega)\right)E_b}{2(2\nu^2+\nu-1)}$$

$$A_{1,2}=\frac{a^{-\frac{\eta}{2}-1}\left(\frac{a}{b}\right)^{\eta}\left(a(\nu-1)\Omega Y_{\frac{\xi-2}{2}}(a\Omega)+(\eta-(\eta+2)\nu)Y_{\frac{\xi}{2}}(a\Omega)-a(\nu-1)\Omega Y_{\frac{\xi+2}{2}}(a\Omega)\right)E_b}{2(2\nu^2+\nu-1)}$$

$$A_{2,1}=\frac{b^{-\frac{\eta}{2}-1}\left(b(\nu-1)\Omega J_{\frac{\xi-2}{2}}(b\Omega)+(\eta-(\eta+2)\nu)J_{\frac{\xi}{2}}(b\Omega)-b(\nu-1)\Omega J_{\frac{\xi+2}{2}}(b\Omega)\right)E_b}{2(2\nu^2+\nu-1)} \tag{13}$$

$$A_{2,2}=\frac{b^{-\frac{\eta}{2}-1}\left(b(\nu-1)\Omega Y_{\frac{\xi-2}{2}}(b\Omega)+(\eta-(\eta+2)\nu)Y_{\frac{\xi}{2}}(b\Omega)-b(\nu-1)\Omega Y_{\frac{\xi+2}{2}}(b\Omega)\right)E_b}{2(2\nu^2+\nu-1)}$$

3. Authentication of the Formulation

To confirm the present dimensionless frequencies, as a first example, a hollow infinite thin-walled cylinder made of an isotropic and homogeneous material is considered. The first ten natural frequencies are listed in Table 1. As seen from this table there is full agreement with those of Gosh's [9] frequencies.

As a second example a hollow infinite cylinder made of a hypothetically power-law graded material is taken into account. Results, which are all based on the similar procedure, are tabulated in Table 2 in a comparative manner. In this table, $\eta = 0$ corresponds a cylinder made of an isotropic and homogeneous material. It is seen from Table 2 that present results are in good harmony with the others. Figure 2 shows also the determinant-dimensionless frequency curve for $\eta = -5$, and *b/a=2*.

Table 1. Comparison of the present non-dimensional results with the open literature for thin-walled cylinder made of an isotropic and homogeneous material ($\nu = 0.3$; $\frac{b}{a} = 1.02$)

	Present	[9]
β_1	0.894602	0.89
β_2	157.082	157.10
β_3	314.161	314.20
β_4	471.24	471.20
β_5	628.319	628.30
β_6	785.399	785.40
β_7	942.478	942.50
β_8	1099.56	1100.00
β_9	1256.64	1257.00
β_{10}	1413.72	1414.00

Table 2. Comparison of the present FGM results with the open literature (*b/a=2*, $\nu = 0.3$).

	$\eta=-5$		$\eta=-2$		$\eta=0$			$\eta=2$		$\eta=5$	
	Present	[37]	Present	[37]	Present	[9]	[37]	Present	[37]	Present	[37]
β_1	0.752017	0.77435	0.684418	0.69407	0.633263	0.6335	0.63563	0.586987	0.58309	0.537493	0.52547
β_2	3.4389	3.43985	3.20038	3.20181	3.21655	3.218	3.21793	3.3714	3.37244	3.81843	3.81886
β_3	6.44741	6.44802	6.31288	6.31356	6.3193	6.319	6.31999	6.40267	6.40331	6.66528	6.66578
β_4	9.53654	9.53698	9.44462	9.44508	9.44864	9.449	9.44910	9.50494	9.50538	9.6854	9.68579
β_5	12.6508	12.65115	12.5813	12.58161	12.5842	12.58	12.58455	12.6266	12.62695	12.7634	12.76368
β_6	15.7758	15.77602	15.7199	15.72016	15.7222	15.72	15.72249	15.7562	15.75647	15.8661	15.86638
β_7	18.9062	18.90638	18.8595	18.85972	18.8614	18.86	18.86165	18.8898	18.89000	18.9816	18.98183
β_8	22.0397	22.03991	21.9997	21.99987	22.0013	22.00	22.00151	22.0256	22.02583	22.1045	22.10466
β_9	25.1753	25.17544	25.1402	25.14037	25.1416	25.14	25.14180	25.1629	25.16309	25.232	25.23214
β_{10}	28.3122	28.31231	28.281	28.28112	28.2822	28.28	28.28239	28.3012	28.30131	28.3626	28.36273

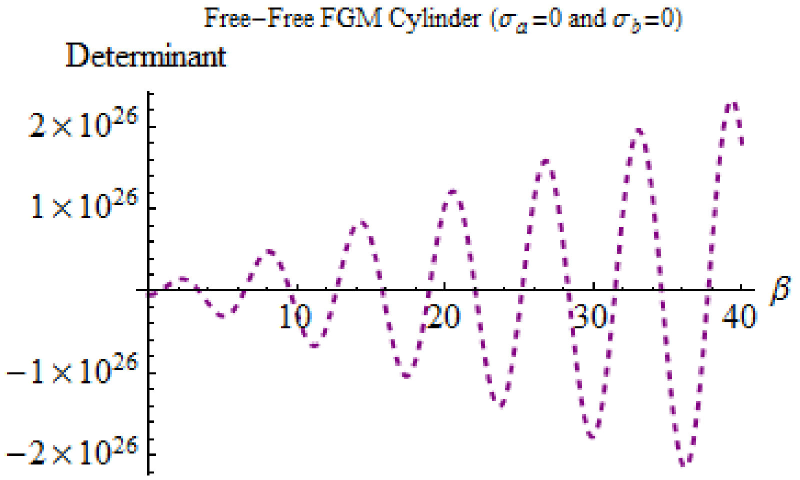

Fig. 2. Determinant-dimensionless frequency curve for $\eta = -5$, and *b/a=2*

4. Effects of the Aspect Ratios and Material Gradients on the Natural Frequencies

In this section a FGM cylinder of $\nu = 0.3$ is considered. The functionally graded material (FGM) is taken to be a hypothetical one exhibiting significant inhomogeneity.

Variation of the natural frequencies with the aspect ratio, which is defined as the ratio of the outer and inner radii, and inhomogeneity indexes is presented in Tables 3 and 4. Table 3 indicates the influence of the aspect ratios and inhomogeneity indexes, which is defined in Eq. (4), on the first ten natural frequencies of a FGM free-free infinite cylinder for $\eta = -5$, $\eta = 0$, $\eta = 7$, and $\eta = 10$. Table 4 shows the influence of the aspect ratios and inhomogeneity indexes on the first three natural frequencies of a FGM free-free infinite cylinder for some inhomogeneity indexes whose values are changed from $\eta = -4$ to $\eta = 5$ with an increase by the unit, except $\eta = 0$.

Table 3. The influence of the aspect ratios and inhomogeneity indexes on the first ten natural frequencies of a FGM free-free infinite cylinder ($\nu = 0.3$) for $\eta = -5$, $\eta = 0$, $\eta = 7$, and $\eta = 10$

b/a	1.02	1.03	1.04	1.05	1.075	1.10	1.25	1.50	1.75	2.00
					$\eta = -5$					
β_1	0.894748	0.890568	0.886514	0.882583	0.873263	0.864622	0.82422	0.785036	0.764088	0.752017
β_2	157.093	104.74	78.566	62.8643	41.9354	31.4778	12.7008	6.49988	4.45582	3.4389
β_3	314.166	209.449	157.093	125.68	83.7996	62.8628	25.2007	12.6787	8.52027	6.44741
β_4	471.243	314.166	235.628	188.506	125.68	94.2684	37.7445	18.9249	12.6628	9.53654
β_5	628.322	418.884	314.166	251.336	167.564	125.679	50.2996	25.1894	16.8278	12.6508
β_6	785.401	523.603	392.704	314.166	209.449	157.092	62.8591	31.4613	21.0022	15.7758
β_7	942.48	628.322	471.243	376.997	251.335	188.506	75.421	37.737	25.1813	18.9062
β_8	1099.56	733.041	549.782	439.828	293.222	219.92	87.9841	44.0147	29.3632	22.0397
β_9	1256.64	837.761	628.322	502.659	335.109	251.335	100.548	50.2939	33.5468	25.1753
β_{10}	1413.72	942.48	706.861	565.49	376.996	282.75	113.112	56.5739	37.7316	28.3122
					$\eta = 0$					
β_1	0.894602	0.890244	0.885946	0.881708	0.871366	0.861367	0.807616	0.736002	0.67955	0.633263
β_2	157.082	104.724	78.5453	62.8386	41.8978	31.4289	12.595	6.33146	4.25189	3.21655
β_3	314.161	209.442	157.082	125.667	83.7808	62.8383	25.147	12.5902	8.40834	6.3193
β_4	471.24	314.161	235.621	188.498	125.667	94.2521	37.7086	18.8654	12.5868	9.44864
β_5	628.319	418.88	314.161	251.329	167.554	125.667	50.2726	25.1446	16.7704	12.5842
β_6	785.399	523.6	392.7	314.161	209.441	157.082	62.8375	31.4254	20.9562	15.7222
β_7	942.478	628.319	471.24	376.992	251.329	188.498	75.403	37.707	25.1429	18.8614
β_8	1099.56	733.039	549.779	439.824	293.217	219.913	87.9687	43.9891	29.3302	22.0013
β_9	1256.64	837.759	628.319	502.656	335.104	251.329	100.535	50.2714	33.5179	25.1416
β_{10}	1413.72	942.478	706.859	565.487	376.992	282.745	113.1	56.5539	37.7059	28.2822
					$\eta = 7$					
β_1	0.894397	0.88979	0.885153	0.880487	0.868724	0.856851	0.785445	0.676595	0.587642	0.517028
β_2	157.133	104.799	78.6447	62.9616	42.0777	31.6628	13.0978	7.11859	5.18815	4.22088
β_3	314.186	209.479	157.132	125.729	83.8709	62.9557	25.4038	13.0114	8.93897	6.92433
β_4	471.257	314.186	235.654	188.539	125.727	94.3304	37.8805	19.1498	12.9492	9.8673
β_5	628.332	418.899	314.185	251.36	167.599	125.726	50.4017	25.3589	17.0445	12.9023
β_6	785.409	523.615	392.72	314.185	209.478	157.129	62.9409	31.5972	21.1763	15.9782
β_7	942.487	628.332	471.256	377.013	251.359	188.537	75.4891	37.8503	25.3268	19.0754
β_8	1099.57	733.05	549.794	439.842	293.243	219.947	88.0425	44.112	29.488	22.1851
β_9	1256.64	837.768	628.332	502.671	335.127	251.358	100.599	50.379	33.6561	25.3026
β_{10}	1413.72	942.487	706.87	565.501	377.012	282.771	113.158	56.6496	37.8288	28.4255
					$\eta = 10$					
β_1	0.89431	0.889597	0.884814	0.879968	0.86761	0.854972	0.777305	0.66029	0.568707	0.498278
β_2	157.178	104.866	78.733	63.0709	42.2372	31.8695	13.5291	7.74807	5.88568	4.9228
β_3	314.209	209.513	157.176	125.783	83.9511	63.06	25.6302	13.3752	9.38644	7.42197
β_4	471.272	314.208	235.684	188.575	125.781	94.4001	38.0328	19.3995	13.2639	10.2267
β_5	628.343	418.916	314.208	251.387	167.639	125.778	50.5163	25.5481	17.2851	13.1796
β_6	785.418	523.628	392.738	314.207	209.51	157.171	63.0327	31.7493	21.3705	16.203
β_7	942.494	628.343	471.271	377.031	251.386	188.572	75.5657	37.9774	25.4894	19.2641
β_8	1099.57	733.059	549.806	439.857	293.265	219.977	88.1082	44.2211	29.6278	22.3475
β_9	1256.65	837.776	628.343	502.685	335.147	251.385	100.657	50.4746	33.7787	25.4452
β_{10}	1413.73	942.494	706.88	565.513	377.03	282.794	113.209	56.7346	37.9379	28.5524

Table 4. The influence of the aspect ratios and inhomogeneity indexes on the first three natural frequencies of a FGM free-free infinite cylinder ($\nu = 0.3$) for $\eta = -4, -3, -2, -1, 1, 2, 3, 4,$ and 5

b/a	1.02	1.03	1.04	1.05	1.075	1.10	1.25	1.50	1.75	2.00
					$\eta = -4$					
β_1	0.894719	0.890503	0.8864	0.882408	0.872884	0.863973	0.820964	0.775819	0.748964	0.731791
β_2	157.088	104.732	78.5558	62.8516	41.9168	31.4536	12.6484	6.41608	4.35369	3.32657
β_3	314.163	209.446	157.088	125.674	83.7903	62.8507	25.1741	12.6348	8.46472	6.38368
					$\eta = -3$					
β_1	0.89469	0.890438	0.886287	0.882233	0.872505	0.863323	0.81766	0.766172	0.732559	0.709067
β_2	157.084	104.726	78.5486	62.8427	41.9037	31.4366	12.6114	6.35679	4.28103	3.24623
β_3	314.161	209.443	157.084	125.669	83.7837	62.8422	25.1554	12.604	8.42558	6.33871
					$\eta = -2$					
β_1	0.89466	0.890373	0.886173	0.882058	0.872125	0.862671	0.814323	0.756219	0.715205	0.684418
β_2	157.082	104.723	78.5444	62.8375	41.8962	31.4268	12.5902	6.32271	4.23937	3.20038
β_3	314.16	209.441	157.082	125.667	83.78	62.8373	25.1447	12.5862	8.40307	6.31288
					$\eta = -1$					
β_1	0.894631	0.890308	0.88606	0.881883	0.871746	0.862019	0.81097	0.746108	0.697358	0.658769
β_2	157.081	104.722	78.5433	62.8362	41.8943	31.4242	12.5847	6.31423	4.22961	3.19044
β_3	314.16	209.441	157.081	125.666	83.779	62.836	25.1419	12.5816	8.39732	6.30639
					$\eta = 1$					
β_1	0.894573	0.890179	0.885833	0.881534	0.870986	0.860715	0.804279	0.726062	0.662312	0.609035
β_2	157.085	104.728	78.5503	62.8448	41.9069	31.4407	12.6209	6.37414	4.30563	3.2776
β_3	314.162	209.443	157.085	125.67	83.7853	62.8443	25.16	12.6119	8.43609	6.35149
					$\eta = 2$					
β_1	0.894543	0.890114	0.885719	0.881359	0.870607	0.860065	0.800975	0.71644	0.646107	0.586987
β_2	157.089	104.734	78.5584	62.8549	41.9216	31.4598	12.6626	6.44176	4.38954	3.3714
β_3	314.164	209.447	157.089	125.675	83.7927	62.8538	25.181	12.6466	8.48039	6.40267
					$\eta = 3$					
β_1	0.894514	0.890049	0.885606	0.881184	0.870228	0.859416	0.797719	0.707266	0.631274	0.567654
β_2	157.095	104.742	78.5695	62.8686	41.9418	31.4861	12.7197	6.5335	4.50181	3.49505
β_3	314.167	209.451	157.094	125.682	83.8028	62.867	25.2099	12.6943	8.54101	6.47243
					$\eta = 4$					
β_1	0.894485	0.889984	0.885492	0.88101	0.86985	0.85877	0.794525	0.698641	0.618006	0.5512
β_2	157.102	104.753	78.5837	62.8862	41.9675	31.5196	12.792	6.64832	4.64025	3.64521
β_3	314.17	209.456	157.102	125.691	83.8156	62.8838	25.2466	12.7548	8.61761	6.56019
					$\eta = 5$					
β_1	0.894456	0.88992	0.885379	0.880835	0.869474	0.858127	0.791408	0.690634	0.606359	0.537493
β_2	157.111	104.766	78.601	62.9076	41.9987	31.5602	12.8794	6.78502	4.80243	3.81843
β_3	314.175	209.463	157.11	125.702	83.8313	62.9041	25.2912	12.828	8.70979	6.66528

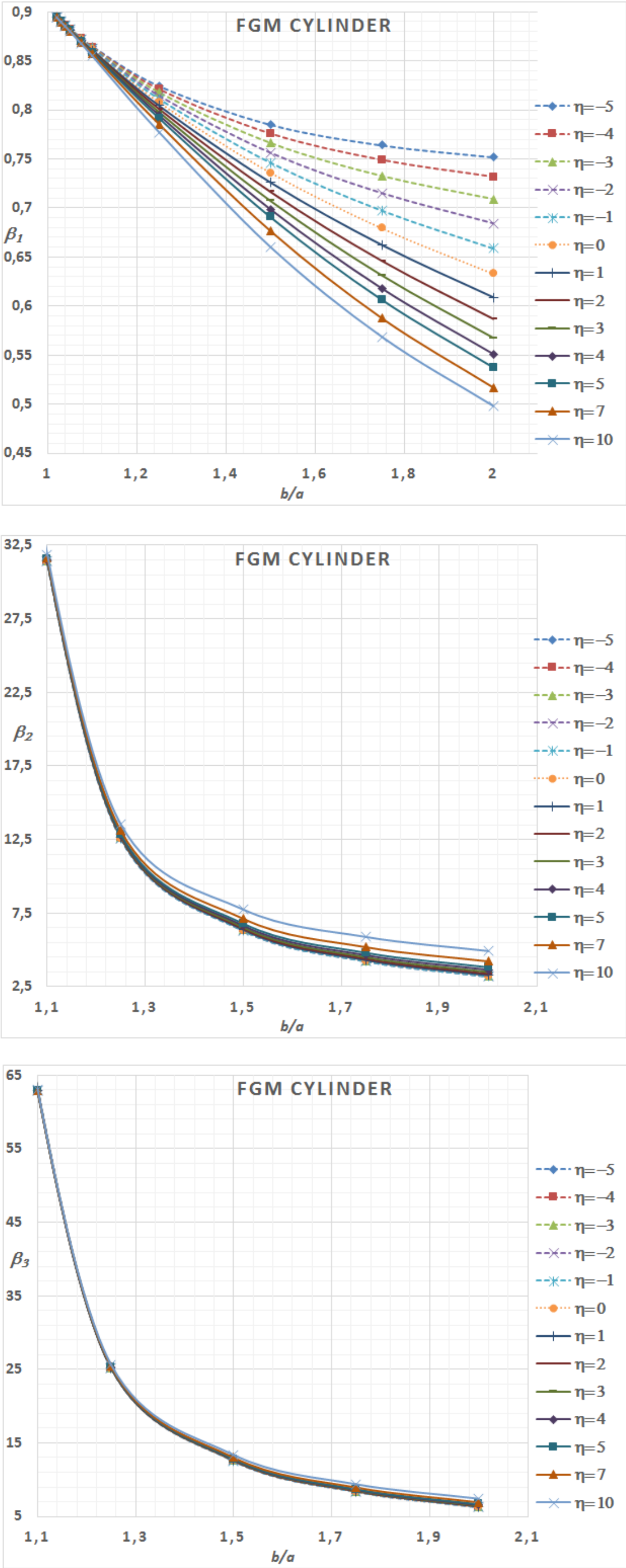

Fig. 3. Variation of the first three natural frequencies with the inhomogeneity indexes for $1.1 \le b/a \le 2$ and $-5 \le \eta \le 10$.

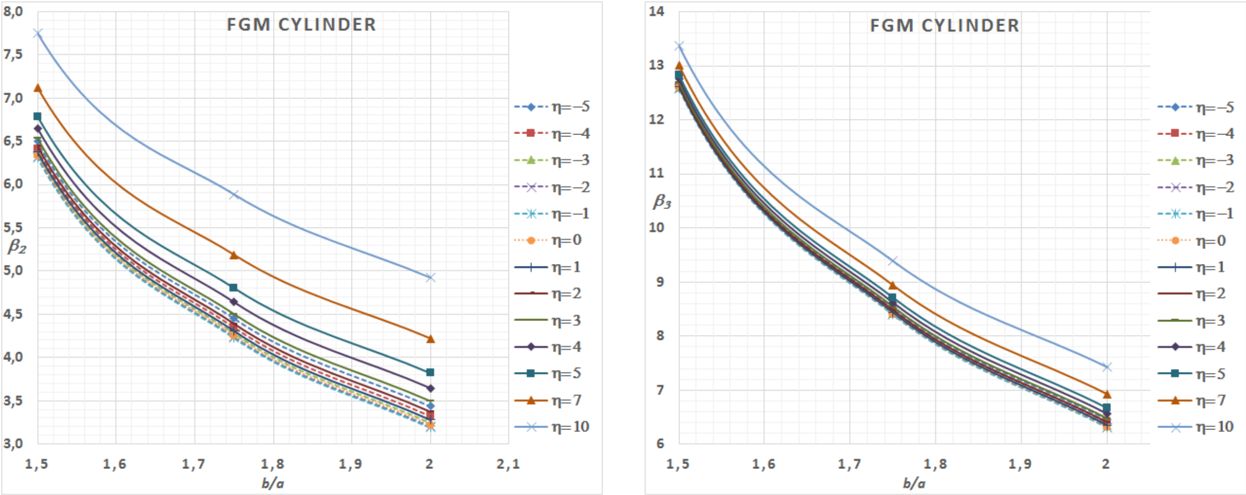

Fig. 4. Variation of the second and the third natural frequencies with the inhomogeneity indexes for $1.5 \leq b/a \leq 2$ and $-5 \leq \eta \leq 10$.

Both Table 3 and Table 4 suggest that as the thickness increases, the dimensionless natural frequencies decrease. This response is slightly observed for fundamental frequencies. However, there is sharply decrease in frequencies of higher modes as the thickness build up.

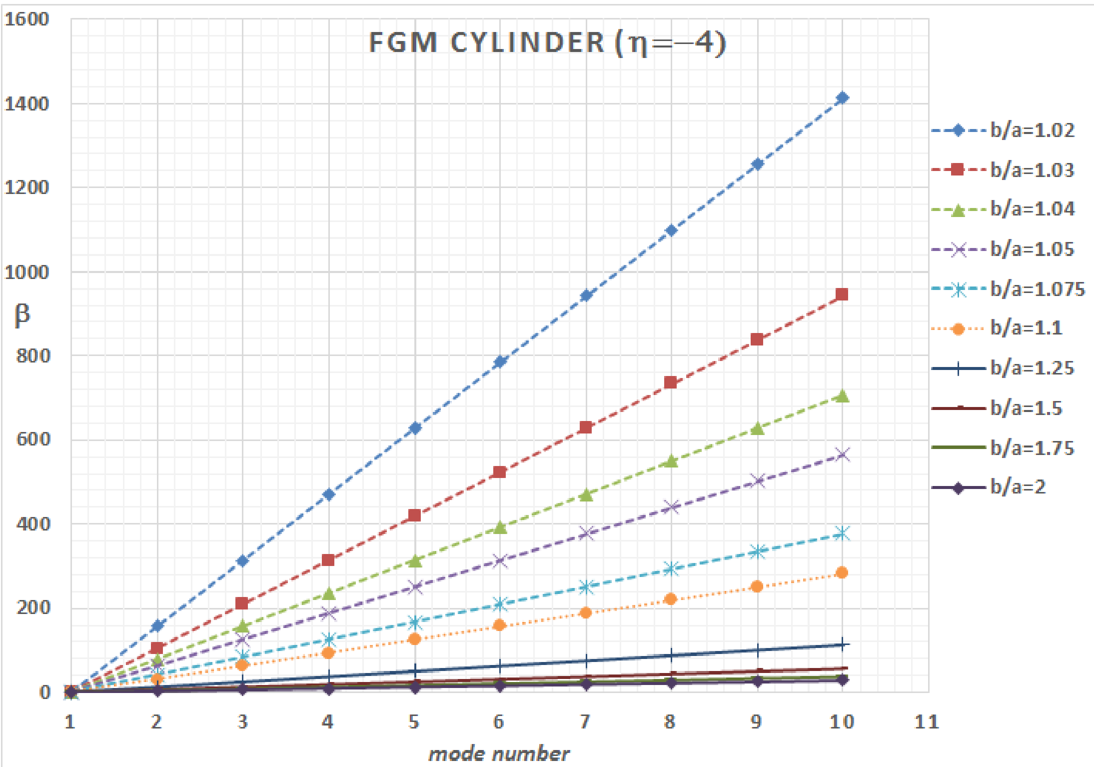

Fig. 5. The relationship between the frequencies of the same cylinder

For the dimensionless frequencies, the degree of to be influenced from the change of inhomogeneity indexes is rather slow. This is demonstrated in Figs. 3 and 4. Figure 3 shows the variation of the first three natural frequencies with the inhomogeneity indexes for $1.1 \leq b/a \leq 2$ and $-5 \leq \eta \leq 10$. Variation of the second and the third natural frequencies with the inhomogeneity indexes for $1.5 \leq b/a \leq 2$ and $-5 \leq \eta \leq 10$ is illustrated in Fig. 4. As seen from Fig. 3, the fundamental frequencies are more affected from the change in the inhomogeneity index. In the interval of the aspect ratio, $1.1 \leq b/a \leq 1.5$, it is almost impossible to observe the variation of natural frequencies in higher modes with the change in the inhomogeneity index. However, as in Fig. 4, this may be clearly observed for $1.5 \leq b/a \leq 2$ for the second and third frequencies.

In this study, it is also revealed that, there is a linear relationship between the pure radial fundamental frequency and others in higher modes of the same cylinder for all inhomogeneity indexes and aspect ratios as seen from Fig. 5.

5. Concluding Remarks

In the present study, the free vibration of hypothetically power-law graded hollow infinite stress-free cylinders is analytically studied to investigate the influences of inhomogeneity indexes and aspect ratios, whose figures are chosen in the broadest possible range of values that can be used in practice, on the natural frequencies. The following conclusions are drawn:

- The fundamental frequency is mainly affected from the variation of the inhomogeneity index than those of other higher frequencies. This may be acceptable as plausible because of huge differences between the fundamental and higher frequencies. For example, for *b/a*=1.02 the natural frequency in the second mode is more than 150 times the fundamental frequency, however it is around four times the first one for *b/a*=2 (see Tables 3-4).
- The effects on the frequencies of the changes in thickness of the cylinder are clearly observed. Increasing thickness produce a substantial decrease in dimensionless frequencies which are particularly in frequencies in higher modes.
- There is also a linear relationship between the frequencies of fundamental and higher modes of the same cylinder.

As stated before, the present results pertaining to the cylinders made of hypothetically FGM materials are obtained by using the same inhomogeneity index for both elasticity modulus and material density. It is a great possibility that there is no such a physical material that provides this feature. For this sense, it may be useful to confirm the present conclusions with physical materials. As mentioned above, this requires the putting numerical solution techniques into practice.

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